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VECTOR-VALUED FUNCTION SPACES

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Preface

The completion of this volume ends a project which has been on our minds for over 20 years. We began the writing of what became the first volume [**137**] in the fall of 1995. It has been a labor of love and we hope it will prove to be a useful addition to the literature. The original idea was to provide a full survey of the work done in characterizing the form of isometries on various Banach spaces. That turned out to be a task too vast in its extent to be done very successfully, but we have tried to touch on as many of its aspects as possible and to include a large number of references. Isometries are of fundamental importance and interest, and results about them pop up in all kinds of places.

The goal of this present volume is the same as in the earlier one, to produce a useful resource for experts in the field, as well as for others who just want to become acquainted with this part of Banach space theory. We have tried to be sensitive to the history of each topic, and so most chapters begin with a discussion of one of the earlier papers. The choice of material to exposit is always difficult, and in most cases we have tried to move through successive developments, so that often the theorems of later sections actually include the earlier ones. We have tried to expose a variety of methods and tools used in the characterizations, even though the end result is often predictable. The notes and remarks are intended to fill in for some of the omissions.

We have relied mostly on the original papers, and most times the proofs are pretty much directly lifted from those sources.

The focus of this volume has been to study isometries on Banach spaces of vector-valued functions, and since we consider it a continuation of [**137**], which we will refer to as Volume 1, we have numbered the chapters from 7 to 13.

Chapter 7 is concerned with the Banach-Stone property, which involves the nature of isometries on the continuous function spaces $C_0(K, X)$, where X is a Banach space. Banach's characterization of isometries on the continuous real-valued functions as weighted composition operators is extended to composition operators with operator weights. The question asks for which Banach spaces X the isometries on $C_0(K, X)$ have this form. We have considered the case where the isometry is from $C_0(Q, X)$ to $C_0(K, Y)$, so that the property involves pairs (X, Y) of spaces. In Chapter 8 a similar question is asked about spaces X for which the isometries on $L^p(\mu, X)$ can be described as a generalization of the form given by Lamperti in the scalar case. Chapter 9 deals with isometries on direct sums of Banach spaces and Hilbert spaces. We also feature a treatment of the case of real scalars, which often presents difficulties in describing isometries. In Chapter 10, we consider isometries on spaces of matrices endowed with a variety of norms. This includes perhaps one of the earliest of all isometry theorems, the theorem of Schur on isometries on the space of $m \times n$ matrices endowed with the operator norm. Chapter 11 is, to some extent, the infinite-dimensional version of Chapter 10. The isometries on the Schatten classes as well as on more general norm ideals of operators are described. In addition, we take a brief look at the noncommutative L^p spaces. In Chapter 12, we change the focus away from vector-valued spaces to consider spaces on which the group of isometries is somehow maximal or else very restricted (minimal). The final chapter treats some topics that are more peripheral and also provides some further references.

Most references are given only in the notes and remarks section of a chapter. We have usually (but not always) attached a name or names to the theorems and some of the lemmas at the time they are stated when that is appropriate. Occasionally there will be a specific reference in the text when a justification for a statement is needed.

As in Volume 1 we assume that the reader is acquainted with the standard material in real and complex analysis and in functional analysis. We have therefore used, without explanation, terminology and notation that are common to those fields. We have put notation in the index if we think it might be unfamiliar or our use may not be standard. Also, we must mention that some notation may be used in different ways in different places. If a notation is being used which is not referenced, then its meaning should be explained within two pages of the point at which it is encountered.

References to items from a chapter numbered from 1 to 6 come from that chapter in Volume 1. Likewise, if a statement is labeled with a number whose first digit is less than 7, it comes from Volume 1. In spite of this, we believe the reader will find the material in this volume mostly self-contained. In fact, the chapters and even sections in the chapters are pretty much independent of each other.

The bibliography is certainly not complete, and there are no doubt some important omissions. However, we hope that the items given, including their own bibliographies, do provide, along with the listed references in Volume 1 and our survey paper [**136**], a good coverage of the literature on isometries.

We would be remiss if we did not mention a few names of individuals who have provided help and encouragement in the preparation of this work. David Sherman provided guidance on the material on noncommutative L^p spaces. Chi-Kwong Li, Pei-Kee Lin, Beata Randrianantoanina, and Geoff Wood answered many questions and helped direct our thinking. Waleed Al-Rawashdeh, David Sherman, Animesh Sarker, and Geoff Wood all read portions of the manuscript making valuable suggestions and correcting errors.

We thank them all. Of course, we alone are responsible for any errors which remain.

We also express our gratitude to Sunil Nair for his patience and encouragement.

This volume is dedicated to Deborah Christine Fleming, April 2, 1962 to June 22, 1993. A daughter, much loved, too briefly.

Richard J. Fleming and James E. Jamison

June 30, 2007

CHAPTER 7

The Banach-Stone Property

7.1. Introduction

In this chapter we return to the problem of characterizing linear isometries from one continuous function space to another. However, unlike in Chapter 2, where we considered continuous scalar-valued functions, we now allow our functions to have values in some Banach space. The goal is to show that an isometry from a space $C_0(Q, X)$ of continuous X-valued functions vanishing at infinity, to a space $C_0(K, Y)$, where Q, K are locally compact Hausdorff spaces and X, Y are Banach spaces, must be given by some version of the canonical form given by Theorem 2.2.1 in Chapter 2. In view of the classical results discussed there, we might expect that the spaces Q and K would necessarily be homeomorphic. However, suppose that Q, K are non-homeomorphic compact Hausdorff spaces and L is a compact Hausdorff space such that $Q \times L$ is homeomorphic to $K \times L$. For example, take

$$Q = \{(a, b) : 1/2 \leq a^2 + b^2 \leq 1; \text{ or } 0 \leq a \leq 2, b = 0\},$$

where (a, b) is a point in the Euclidean plane,

$$K = \{(a, b) : 1/2 \leq a^2 + b^2 \leq 1; \text{ or } 1 \leq a \leq 2, b = 0; \text{ or } a = 0, 1 \leq b \leq 2\},$$

and L is the unit interval. Then for any Banach space X, $C(L \times Q, X)$ is isometric to $C(L \times K, X)$ and so $C(Q, C(L, X))$ is isometric with $C(K, C(L, X))$. As a result of all this, we can expect that any extension of the Banach-Stone theorem to continuous functions with values in Banach spaces X, Y will require some special conditions on these spaces.

The natural extension of the canonical form of Chapter 2, that is, the writing of an isometry as a "weighted composition operator," would be to write the operator as a composition with operator weights. Thus for an isometry T from $C_0(Q, X)$ to $C_0(K, Y)$ we look for T to be of the form

$$(1) \qquad TF(t) = V(t)F(\varphi(t)) \text{ for all } t \in K,$$

where φ is a function from K to Q with certain nice properties, and $t :\rightarrow V(t)$ describes some kind of function from K into the space of bounded operators from X into Y.

Several questions present themselves.

(i) What conditions on X and Y will force T to be of the desired form?

(ii) Are there conditions under which a space Y is universal in the sense that T will be of the desired form for any X?
(iii) What can be said in the case where $X = Y$?
(iv) Are there significant differences in the surjective and nonsurjective cases for the isometry T?
(v) What are the conditions on φ and $V(t)$ which we expect to be satisfied?

The last question is important, because we might be tempted to require φ to be a homeomorphism and each $V(t)$ to be an isometry. Yet, consider the following example. Recall that by $\ell^p(n)$ we mean the space of n tuples of scalars with the usual p norm.

7.1.1. EXAMPLE. *Let T be defined on $C(\{1\}, \ell^1(2))$ to $C(\{1,2\}, \mathbb{R})$ by*

$$TF(t) = V(t)F(\varphi(t)),$$

where φ is identically 1 and $V(1)$, $V(2)$ are the linear functionals on $\ell^1(2)$ determined by the vectors $(1,1)$ and $(1,-1)$, respectively. Then T is an isometry from $C(Q,X)$ onto $C(K,Y)$, where Q, K are not homeomorphic, and X, Y are not isometric.

Let us set the stage for our investigations by means of a formal definition.

7.1.2. DEFINITION.

(i) *A pair (X,Y) of Banach spaces will be said to satisfy the* Banach-Stone property *if for every surjective isometry T from $C_0(Q,X)$ to $C_0(K,Y)$, where Q, K are locally compact Hausdorff spaces, there is a continuous function φ from K onto Q and a map $t :\to V(t)$ which is continuous from K into the space $\mathcal{L}(X,Y)$ of bounded operators from X to Y with the strong operator topology such that*

$$TF(t) = V(t)F(\varphi(t)) \;\; \textit{for all} \;\; t \in K.$$

(ii) *If, in (i) above, φ is a homeomorphism and each $V(t)$ is an isometry, we say that (X,Y) has the* strong Banach-Stone property.
(iii) *If all that can be said is that Q and K are homeomorphic, we say that (X,Y) has the* weak Banach-Stone property.
(iv) *If (X,X) has any of the properties above, we will say that X has the property.*

In this context, the work in Chapter 2 shows that both $\mathbb{R}$ and $\mathbb{C}$ have the strong Banach-Stone property. (Note, as usual, when X is the scalar field, we write $C_0(Q,X) = C_0(Q)$.)

In Section 2, we will show that a strictly convex space has the strong Banach-Stone property. The work of Jerison discussed there was the first attempt to extend the Banach-Stone theorem to the vector-valued case. We will see that a strictly convex space is universal for the Banach-Stone property in the sense that if Y is strictly convex, then (X,Y) has the property for any Banach space X. Jerison established a condition on X which guaranteed that it had the weak Banach-Stone property. Our discussion there also includes some contributions of Cambern, including his identification of a property (called property (P)) equivalent to the Banach-Stone property.

Cambern ultimately showed that any space which is reflexive and contains no M summand must satisfy his property (P). He also gave an example of an isometry that does not have the canonical form. These and other matters are covered in Section 3.

In Section 4 we consider (in an abbreviated form) the M structure theory of Behrends. Here we show that a space Y with trivial centralizer is universal for the Banach-Stone property and that any pair of spaces (X, Y) has the strong Banach-Stone property if both X and Y have trivial centralizers. These results contain those of previous sections as well as other similar theorems appearing in the literature.

The case where the isometry T from $C_0(Q, X)$ to $C_0(K, Y)$ is not surjective is treated in Section 5. Cambern's extension of Holsztynski's theorem to vector-valued functions is featured here as well as Font's generalizations to nonsurjective isometries defined on subspaces M of $C_0(Q, X)$. In this section it is always assumed that Y is strictly convex.

Section 6 is devoted to some work involving characterizations of *nice* operators, which were introduced in Section 4. An operator is nice if its conjugate maps the extreme points of the dual unit ball (of the range of the operator) to extreme points of the dual unit ball of the domain. An isometry is a nice operator but there are nice operators which are not isometries. This concept lends itself to the use of extreme point techniques for nice operators on $C_0(Q, X)$ to $C_0(K, Y)$. Following the same line of investigation as in Section 3 of Chapter 2, we also consider operators from subspaces M of $C_0(Q, X)$ onto subspaces N of $C_0(K, Y)$. This general setting is fraught with difficulties, not only for nice operators, but for isometries as well. We adapt the M-structure approach to this situation, too. We dispense with the assumption that Y is strictly convex. This requires us to put some other conditions on the range space N.

We mention here that throughout the sequel, $B(X)$ will denote the closed unit ball of the space X and $S(X)$ will denote its surface.

7.2. Strictly Convex Spaces and Jerison's Theorem

We intend to show in this section that a strictly convex Banach space satisfies the strong Banach-Stone property. The key to writing an operator T as a weighted composition is to establish the pairing between $t \in K$ and $s \in Q$ so that the function given by $s = \varphi(t)$ is well defined. In this section, the pairing is accomplished by making use of the concept of a T-set.

7.2.1. Definition. *A T-set is a subset S of a Banach space X with the property that for any finite collection, $x_1, x_2, \ldots, x_n$ of elements of S,*

$$\|\sum_{j=1}^{n} x_j\| = \sum_{j=1}^{n} \|x_j\|,$$

and such that S is maximal with respect to this property.

For our purposes, the following characterization of T-sets in $C_0(K, X)$ will be crucial.

7.2.2. Lemma. *For any T-set S in the Banach space X and any t in the locally compact space K, the set*

$$(S, t) = \{F \in C_0(K, X) : F(t) \in S, \text{ and } \|F(t)\| = \|F\|\} \tag{2}$$

is a T-set in $C_0(K, X)$. Conversely, any T-set in $C_0(K, X)$ is of the form given in (2).

Proof. It is straightforward to show that the set (S, t) has the norm additive property. To show it is maximal, we assume that $H \in C_0(K, X)$ has the property that $\|H + F\| = \|H\| + \|F\|$ for all $F \in (S, t)$. We must show that $H \in (S, t)$.

Suppose $\|H(t)\| < \|H\|$. Let $D = \{s \in K : \|H(s)\| = \|H\|\}$. Then D is a nonempty closed subset of K with $t \notin D$ and so there exists $h \in C_0(K)$ with $h(t) = 1$ and $h(s) = 0$ for all $s \in D$. Let u be a nonzero element of S and define G to be the function $h(\cdot)u$. Then $G \in (S, t)$ and it is easy to see that

$$\|G(s) + H(s)\| \leq \|G(s)\| + \|H(s)\| < \|G\| + \|H\|.$$

Since there must be some $s \in K$ for which $\|(G + H)(s)\| = \|G + H\|$, we have

$$\|G + H\| < \|G\| + \|H\|,$$

which contradicts the choice of H.

Suppose $H(t) \notin S$. Then there exists, by the maximality of S, some $u \in S$ such that $\|H(t) + u\| < \|H(t)\| + \|u\|$. Let

$$U = \{s \in K : \|H(s) + u\| < \|H(s)\| + \|u\|\}.$$

Then U is an open neighborhood of t and there exists $g \in C_0(K)$ such that $g(t) = 1$ and $g(s) = 0$ for all $s \in K \backslash U$. If $G = g(\cdot)u$, then $G \in (S, t)$, and we have

$$\|G(s) + H(s)\| = \|[1 - g(s)]H(s) + g(s)[u + H(s)]\| < \|H\| + \|G\|$$

for all $S \in K$. This leads to a contradiction of the choice of H as above.

To prove the converse, suppose that $\mathcal{S}$ is a T-set. For each $F \in \mathcal{S}$ let $K_F = \{s \in K : \|F(s)\| = \|F\|\}$. Then K_F is closed and nonempty for each F and, in fact, since F vanishes at infinity, K_F is compact. If $F_1, F_2, \ldots, F_n$ is any finite collection of elements of $\mathcal{S}$, then $\cap_{j=1}^{n} K_{F_j} \neq \emptyset$. For if it were empty, then for any $s \in K$, there would exist some $F_j \in \mathcal{S}$ such that $\|F_j(s)\| < \|F_j\|$. This would imply that

$$\|\sum_{j=1}^{n} F_j(s)\| \leq \sum_{j=1}^{n} \|F_j(s)\| < \sum_{j=1}^{n} \|F_j\|$$

for all $s \in K$. It will now follow that

$$\|\sum_{j=1}^{n} F_j\| < \sum_{j=1}^{n} \|F_j\|,$$

which contradicts the T-set property of $\mathcal{S}$. Therefore, because K is locally compact, and the sets K_F are compact, there must be some $t \in \cap_{F \in \mathcal{S}} K_F$. It can be shown that the set $S_0 = \{F(t) : F \in \mathcal{S}\}$ has the additive property of a T-set, and by Zorn's lemma, we may find a maximal set S with this property containing S_0. Then S is a T-set in X, and by the first part of the proof, the set (S, t) is a T-set in $C_0(K, X)$ which contains $\mathcal{S}$. But $\mathcal{S}$ is maximal, and we conclude that $\mathcal{S} = (S, t)$. □

We will say that a T-set (S, s) is *based* at s.

It is clear that an isometry must map a T-set to a T-set. We observe that if an isometry T has the canonical form

$$TF(t) = V(t)F(\varphi(t))$$

where $V(t)$ is a bounded operator and φ is a continuous function, then given a T-set (R, t) in $C_0(K, Y)$ as in the previous lemma, the T-set, $T^{-1}(R, t)$, must be based at $s = \varphi(t)$. Otherwise, if $T^{-1}(R, t) = (S, r)$ where $r \neq s$, then we may choose a nonzero $F \in (S, r)$ such that $F(s) = 0$. Then $TF \in (R, t)$, and

$$0 \neq \|F\| = \|TF\| = \|TF(t)\| = \|V(t)F(\varphi(t))\| = 0.$$

7.2.3. Definition. *A pair (X, Y) of Banach spaces will be said to satisfy property (P) if given an isometry T from $C_0(Q, X)$ onto $C_0(K, Y)$, and $t \in K$, the T-sets in $C_0(Q, X)$ given by $\{T^{-1}(R, t)$: R a T-set in $Y\}$ are all based at a single point $s \in Q$.*

It follows then from our remark prior to the definition that if (X, Y) has the BSP, then (X, Y) has property (P). We want to show now that the converse of that statement holds. First, we need some lemmas. By $ext(M)$ we mean the extreme points of the set M. If Y is a Banach space, then $ext(Y)$ will be the extreme points of the unit ball of Y. Indeed, we may speak of the extreme points of a Banach space Y, but we mean the extreme points of the unit ball of Y.

7.2.4. Lemma. *If S is a T-set in a Banach space X, there exists $x^* \in ext(X^*)$ such that $x^*(x) = \|x\|$ for all $x \in S$.*

Proof. Let

$$\Gamma(S) = \{z^* \in X^* : \|z^*\| = 1, z^*(x) = \|x\| \text{ for all } x \in S\}.$$

For each $x \in X$, let $M_x = \{y^* \in B(X^*) : y^*(x) = \|x\|\}$. Then each M_x is a nonempty weak*-closed subset of the weak*-compact unit sphere of X^*. Because of the norm additive property of the T-set S, the collection $\{M_x : x \in S\}$ has the finite intersection property and so the entire collection has nonempty intersection. An element of that intersection is a member of $\Gamma(S)$. Now $\Gamma(S)$ is nonempty, convex, and weak*-closed, and so must have an extreme point x^* by the Krein-Milman theorem. We claim that x^* is also an extreme point of $B(X^*)$.

Suppose $x^* = (1/2)(y^* + z^*)$, where $y^*, z^* \in B(X^*)$. Given $x \in S$, we have

$$\|x\| = x^*(x) = \frac{y^*(x)}{2} + \frac{z^*(x)}{2},$$

from which we conclude that $|y^*(x)| = |z^*(x)| = \|x\|$. Thus $y^*(x)$, $z^*(x)$, and $x^*(x)$ are all scalars on the circle of radius $\|x\|$, and since one is a convex combination of the others, they must all be the same scalar. Hence, y^* and z^* are both in $\Gamma(S)$, and since $x^* \in ext(\Gamma(S))$, we must have $x^* = y^* = z^*$. □

We want to use the notation $\Gamma(S)$ also in the next lemma. It will also be convenient later to have a designation for elements such as are guaranteed by Lemma 7.2.4. We will denote by $\mathfrak{F}$ the set of all $y^* \in ext(Y^*)$ such that $y^* \in \Gamma(S)$ for some T-set S in Y. As in earlier chapters, we will write ψ_t for the evaluation functional, $\psi_t(F) = F(t)$.

7.2.5. LEMMA. *Suppose T is an isometry from $C_0(Q, X)$ onto $C_0(K, Y)$. If (S, s) is a T-set in $C_0(Q, X)$ and $T(S, s) = (R, t)$, then for any $y^* \in ext(Y^*) \cap \Gamma(R)$, there is $x^* \in ext(X^*) \cap \Gamma(S)$ such that*

$$(T^*)(y^* \circ \psi_t) = x^* \circ \psi_s. \tag{3}$$

PROOF. Suppose $T(S, s) = (R, t)$ as in the hypotheses, and let y^* be an extreme point of $B(Y^*)$ as guaranteed by Lemma 7.2.4. Then, by Corollary 2.3.6 of Chapter 2, $y^* \circ \psi_t$ is an extreme point of the dual unit ball of $C_0(K, Y)$. Since (T^*) is an isometry, we have

$$(T^*)(y^* \circ \psi_t) = x^* \circ \psi_r$$

for some $r \in Q$ and $x^* \in ext(X^*)$. Let $u \in R$ with $\|u\| = 1$ and let $g \in C_0(K)$ with $\|g\| = g(t) = 1$. Then $G = g(\cdot)u$ is an element of the T-set (R, t), so that $F = T^{-1}G \in (S, s)$. Suppose $r \neq s$ and let $x = F(s)$. Further, let $H = h(\cdot)x$, where h is a function in $C_0(Q)$ with the property that $\|h\| = h(t) = 1$ and $h(r) = 0$. It follows that $H \in (S, s)$ so that $TH \in (R, t)$. Now we must have

$$\begin{aligned} \|u\| = \|x\| = \|H\| = \|TH\| &= \|TH(t)\| = |y^*(TH(t))| \\ &= |T^*(y^* \circ \psi_t)(H)| \\ &= |x^*(H(r)| \\ &= 0. \end{aligned}$$

This contradiction means that $r = s$, so that (3) is satisfied. It remains to observe that $x^* \in \Gamma(S)$. For this, let $x \in S$ and choose a norm 1 function $f \in C_0(Q)$ such that $f(s) = 1$. Then $F = f(\cdot)x \in (S, s)$ so that $TF \in (R, t)$. It is straightforward to verify that

$$x^*(x) = y^*(TF(t) = \|TF(t)\| = \|TF\| = \|F\| = \|x\|.$$

□

One of the properties satisfied by a weighted composition operator T is that if $F(s) = 0$ where $s = \varphi(t)$, then $TF(t) = 0$. This property can also be valuable in establishing the canonical form as will be seen when we prove the main theorem of this section. The next lemma indicates how the property arises.

7.2.6. Lemma. *Suppose that (X, Y) satisfies property (P), and T is an isometry from $C_0(Q, X)$ onto $C_0(K, Y)$. If s, t are related as in the definition of property (P), and $F(s) = 0$, where $F \in C_0(Q, X)$, then $TF(t) = 0$.*

Proof. Suppose $F(s) = 0$ and $TF(t) = u$. Let R be a T-set containing u and let (S, s) be a T-set such that $T(S, s) = (R, t)$. Let $y^* \in \Gamma(R) \cap ext(Y^*)$. By Lemma 7.2.5 there is $x^* \in ext(X^*)$ such that (3) holds. Then

$$\begin{aligned} \|u\| = y^*(u) = y^*(TF(t)) &= T^*(y^* \circ \psi_t)(F) \\ &= (x^* \circ \psi_s)(F) \\ &= x^*(F(s)) = 0. \end{aligned}$$

We conclude that $TF(t) = 0$. □

7.2.7. Theorem. *A pair (X, Y) of Banach spaces has the Banach-Stone property if and only if (X, Y) has property (P).*

Proof. Assume (X, Y) has property (P) and that T is an isometry from $C_0(Q, X)$ onto $C_0(K, Y)$. For each t in K, let $\varphi(t) = s$ denote the element of Q guaranteed by property (P). Then φ is well defined and clearly onto all of Q. Define the operator $V(t)$ from X to Y by $V(t)x = TF(t)$, where $F \in C_0(Q, X)$ and $F(\varphi(t)) = F(s) = x$. Then $V(t)$ is well defined, for if H is another function in $C_0(Q, X)$ with $H(s) = x = F(s)$, we have $(H - F)(s) = 0$. By Lemma 7.2.6, we must have $T(H - F)(t) = 0$, or $TH(t) = TF(t)$. Furthermore, if $x \in X$, let S be a T-set containing x and $F \in (S, s)$ where $F(s) = x$. Then

$$\|V(t)x\| = \|TF(t)\| \le \|TF\| = \|F\| = \|x\|$$

and we see that $V(t)$ is bounded for each $t \in K$.

Next we show that φ is continuous. If φ is not continuous at $t \in K$, there is a net $\{t_\beta\}$ in K such that $t_\beta \to t$ but $\varphi(t_\beta) \nrightarrow \varphi(t)$. Hence there is a neighborhood U of $\varphi(t) = s$ in Q so that for every β_0 there is some $\beta \ge \beta_0$ such that $\varphi(t_\beta) \notin U$. For any $y^* \in \mathfrak{F}$, we have that $y^* \circ \psi_{t_\beta} \to y^* \circ \psi_t$ in the weak*-topology, so that $T^*(y^* \circ \psi_{t_\beta}) \to T^*(y^* \circ \psi_t)$ in the weak*-topology as well. If $T^*(y^* \circ \psi_t) = x^* \circ \psi_s$, we may choose $x \in X$ so that $x^*(x) \ne 0$.

There exists $f \in C_0(Q)$ such that $f(s) = 1$ and $f(r) = 0$ for $r \in Q \backslash U$. Let $F = f(\cdot)x \in C_0(Q, X)$. Now for any β_0, there is some $\beta \geq \beta_0$ such that $\varphi(t_\beta) \notin U$ and so $T^*(y^* \circ \psi_{t_\beta})(F) = 0$. But $T^*(y^* \circ \psi_t)(F) \neq 0$ so we cannot have $T^*(y^* \circ \psi_{t_\beta})(F)$ converging to $T^*(y^* \circ \psi_t)(F)$, which is a contradiction to the weak*-convergence mentioned above. We conclude that φ must be continuous.

To complete the proof that (X, Y) has BSP, we must establish the continuity of the map $t :\rightarrow V(t)$, where the range space is given the strong operator topology (S.O.T.). Given a net $\{t_\alpha\}$ which converges to t in K, we must show that $V(t_\alpha)x \rightarrow V(t)x$ for each $x \in X$. Let $\epsilon > 0$ be given. We may choose a compact neighborhood U of $\varphi(t)$ in Q and a function $f \in C_0(Q)$ such that $f(r) = 1$ for every $r \in U$. By the continuity of φ, there exists a neighborhood U_0 of t such that $\varphi(U_0) \subset U$. Let $x \in X$ and let $F = f(\cdot)x$. Since TF is continuous at t, there exists α_0 such that

$$\|TF(t_\alpha) - TF(t)\| < \epsilon$$

for all $\alpha \geq \alpha_0$. Furthermore, since $\{t_\alpha\} \rightarrow t$, we may choose α_0 such that $t_\alpha \in U_0$ for $\alpha \geq \alpha_0$. Hence, for such α, we have

$$\|V(t_\alpha)x - V(t)x\| = \|TF(t_\alpha) - TF(t)\| < \epsilon.$$

For the proof of the "only if" statement, we refer to the remark prior to Lemma 7.2.4. □

7.2.8. Corollary. *The pair (X, Y) has the strong Banach-Stone property if and only if both (X, Y) and (Y, X) have property (P).*

Proof. Apply Theorem 7.2.7 to both T and T^{-1} to get φ one-to-one and continuous both ways. Given $t \in K$, and $x \in X$, let S be a T-set in X containing x, and choose $F \in (S, s)$, where $s = \varphi(t)$. Since (Y, X) has property (P), $T(S, s)$ must be of the form (R, t) for some T-set R containing $TF(t)$ and such that $\|TF(t)\| = \|TF\|$. Hence, we have

$$\|V(t)x\| = \|TF(t)\| = \|TF\| = \|F\| = \|x\|$$

and we conclude that $V(t)$ is an isometry. □

7.2.9. Corollary. *(Cambern) A Banach space has the strong Banach-Stone property if and only if it satisfies property (P).*

We now look for a property for a Banach space that will imply that it has property (P). We need another definition.

7.2.10. Definition.

(i) *Two T-sets S and R in a Banach space X are said to be* discrepant *if either $R \cap S = \{0\}$, or there exists a T-set L such that $R \cap L = S \cap L = \{0\}$.*
(ii) *A Banach space X is said to satisfy property (D) if any two T-sets in X are discrepant.*

7.2.11. EXAMPLE.

(i) *In the plane with a parallelogram for the unit sphere, there are four T-sets, one for each side of the parallelogram. Opposite T-sets are discrepant, while adjacent ones are not.*

(ii) *The plane with any unit sphere other than the parallelogram has the property that any two T-sets are discrepant.*

(iii) *In a three-dimensional space with the portion of a circular cylinder between two planes taken as unit sphere, there are two kinds of T-sets. One kind consists of the nonnegative multiples of the points on a face of the cylinder, and the other of the nonnegative multiples of the points of a generator. Two T-sets of the same kind are discrepant, while one of one kind and one of the other kind are not.*

7.2.12. LEMMA. *If (S,s) and (R,r) are T-sets in $C_0(Q,X)$, and $(S,s)\cap(R,r)=\{0\}$, then $S\cap R=\{0\}$ and $r=s$.*

PROOF. Suppose $x\in S\cap R$ with $x\neq 0$. There exists $f\in C_0(Q)$ with $f(r)=f(s)=1=\|f\|$. The function $F=f(\cdot)x$ is in $(S,s)\cap(R,r)$.

Next suppose $r\neq s$. Let U_r, U_s be neighborhoods of r,s, respectively, such that $U_r\cap U_s=\emptyset$. If f,g are functions in $C_0(Q,X)$ so that $f(r)=1$, $f(Q\backslash U_r)=0$, and $g(s)=1$, $g(Q\backslash U_s)=0$, then for nonzero elements $x\in R$ and $y\in S$, the function $F=f(\cdot)x+g(\cdot)y$ will be a nonzero element of $(S,s)\cap(R,r)$. □

7.2.13. THEOREM. *If Y is a Banach space that satisfies property (D), then the pair (X,Y) satisfies property (P) for any Banach space X. Hence, (X,Y) also satisfies the Banach-Stone property for any Banach space X.*

PROOF. Let R, R_1 be T-sets in Y and suppose $T^{-1}(R,t)=(S,s)$ and also that $T^{-1}(R_1,t)=(S_1,r)$ for some $t\in K$, where T is an isometry from $C_0(Q,X)$ onto $C_0(K,Y)$. Since Y has property (D), the T-sets R and R_1 are discrepant. It follows easily that the T-sets (R,t) and (R_1,t) are also discrepant. Furthermore, it is straightforward to show that an isometry preserves discrepancy, so that (S,s) and (S_1,r) are discrepant as well. If their intersection is $\{0\}$, then $r=s$ by Lemma 7.2.12. If, instead, there is a T-set (L,q) such that

$$(S,s)\cap(L,q)=\{0\}=(L,q)\cap(S_1,r),$$

then again by Lemma 7.2.12 we have $s=q$ and $q=r$ so that $s=r$. We conclude that (X,Y) has property (P). The second statement follows from Theorem 7.2.7. □

7.2.14. COROLLARY. *If both X and Y have property (D), then (X,Y) has the strong Banach-Stone property.*

PROOF. Since both X and Y have property (D), then both (X,Y) and (Y,X) have property (P) and so (X,Y) has the SBSP by Corollary 7.2.8. □

7.2.15. Corollary. *(Jerison) If X has property (D), then X satisfies the strong Banach-Stone property.*

Because a strictly convex space has the property that any two T-sets (which are rays from the origin) intersect only in the zero element, we have the following theorem.

7.2.16. Theorem. *If Y is a strictly convex Banach space, then (X, Y) has the Banach-Stone property for any Banach space X. If both X and Y are strictly convex, then (X, Y) has the strong Banach-Stone property.*

Proof. A strictly convex space satisfies property (D). The theorem follows from Theorems 7.2.13 and 7.2.14. □

Because of its historical significance, we want to state specifically the theorem of Jerison.

7.2.17. Theorem. *(Jerison) If X is strictly convex, then X satisfies the strong Banach-Stone property.*

7.3. M Summands and Cambern's Theorem

We search now for a wider class of spaces that might satisfy property (P), and therefore the Banach-Stone property. We begin with an example that shows us where not to search.

7.3.1. Example. *Let Y be a Banach space, and suppose M, N are closed subspaces of Y with $M \cap N = \{0\}$. Suppose Y can be written in the form*

$$Y = M \oplus_\infty N,$$

where the notation indicates that $y = u + v$ for each $y \in Y$ and some $u \in M, v \in N$ and also that $\|y\| = max\{\|u\|, \|v\|\}$. Let K be the discrete space $\{1, 2\}$, and consider the Banach space $C(K, Y)$. Each $F \in C(K, Y)$ can be written as a sum, $F = F_1 + F_2$ where $F_1(k) \in M$ and $F_2(k) \in N$ for $k = 1, 2$. Define T on $C(K, Y)$ onto itself by

$$TF(1) = F_1(1) + F_2(2); \quad TF(2) = F_1(2) + F_2(1).$$

It is clear that for an F taking its norm only at 1, TF might take on its norm only at 1, while for another such F, TF might take on its norm only at 2. Property (P) is not satisfied by Y.

We recall that a subspace M of a Banach space Y as in the example above is called an *M summand* of Y. Hence the example shows that any space Y which has a nontrivial M summand cannot satisfy property (P) and therefore cannot have the strong Banach-Stone property.

It is natural to ask whether a space with no nontrivial M summand must have the SBSP. We are going to prove that the answer is yes, at least for reflexive spaces.

Recall that $\mathfrak{F}$ denotes the collection of all $y^* \in \Gamma(S) \cap ext(Y^*)$ for some T-set S in Y. For a given set A we will let $sp(A)$ denote the linear span of A and $\overline{sp}(A)$ the closed linear span of A.

7.3.2. Lemma. *Let Y be a reflexive Banach space. Suppose there exists a Hamel basis $H \subset \mathfrak{F}$ for $sp(\mathfrak{F}) \subset Y^*$, which can be partitioned into two nonempty subsets S_1 and S_2 such that $\overline{sp}(S_1) \cap \overline{sp}(S_2) = \{0\}$. If it is true that for any $y^* \in \mathfrak{F}$, either $y^* \in sp(S_1)$ or $y^* \in sp(S_2)$, then Y has a nontrivial M-summand.*

Proof. Let M be the annihilator of S_2 in $Y^{**} = Y$ and let N be the annihilator of S_1. If u is a nonzero element of $M \cap N$, and $y^* \in \mathfrak{F}$, then it is clear that $y^*(u) = 0$. However, there must be a T-set S containing u and therefore some $y^* \in \mathfrak{F}$ such that $y^*(u) = \|u\| \neq 0$. Hence, we must have $M \cap N = \{0\}$. Let $Y_0 = M \oplus N$. We want to show that Y_0 is closed.

Suppose $u \in M$ with $\|u\| = 1$, and let $v \in N$. Since u is in some T-set, there exists $y^* \in \mathfrak{F}$ such that $y^*(u) = \|u\| = 1$. If $y^* \in sp(S_2)$, then $y^*(u) = 0$, so by the hypothesis, we conclude that $y^* \in sp(S_1)$. Thus

$$\|u + v\| \geq |y^*(u + v)| = |y^*(u)| = \|u\| = 1,$$

so that we have shown that $\|u + v\| \geq 1$ for all such u, v. It follows that for any $u \in M$, $v \in N$ we have

$$\|u + v\| \geq \|u\|.$$

Using this and a form of the usual triangle inequality, we obtain

$$\|u + v\| \leq \|u\| + \|v\| \leq 2\|u\| + \|u + v\| \leq 3\|u + v\|.$$

This inequality shows that the usual norm on Y_0 is equivalent with the norm $\|u+v\|_1 = \|u\|+\|v\|$. Since the space is complete in the latter norm, it follows that Y_0 is closed in the given norm. Suppose that $Y_0 \neq Y$. In this case, there must exist some nonzero z^* such that z^* annihilates Y_0. This would mean that $z^* \in M^0 = \overline{sp}(S_2)$, and also $z^* \in N^0 = \overline{sp}(S_1)$. This is impossible since $\overline{sp}(S_1) \cap \overline{sp}(S_2) = \{0\}$. Hence, we have $Y = M \oplus N$.

Finally we must show that if $y = u + v$ for $u \in M$, $v \in N$, then $\|y\| = max\{\|u\|, \|v\|\}$. Given such $y \neq 0$, there exists $y^* \in \mathfrak{F}$ such that $y^*(y) = \|y\|$. If $y^* \in sp(S_1)$, then $y^*(v) = 0$ and

$$\|y\| = y^*(y) = y^*(u) + y^*(v) = y^*(u) \leq \|u\|.$$

Similarly, there exists $z^* \in \mathfrak{F}$ such that $z^*(u) = \|u\|$. Now z^* cannot be in $sp(S_2)$, for if so, then $0 = z^*(u) = \|u\| \geq \|y\| > 0$. Hence, $z^* \in sp(S_1)$ and

$$\|y\| \geq |z^*(y)| = |z^*(u + v)| = |z^*(u)| = \|u\|.$$

The two displayed inequalities show that $\|y\| = \|u\|$. On the other hand, there exists $w^* \in \mathfrak{F}$ such that $w^*(v) = \|v\|$. If $w^* \in sp(S_1)$, then $w^*(v) = 0$ so that

$$\|y\| = \|u\| = max\{\|u\|, \|v\|\}.$$

If $w^* \in sp(S_2)$, then

$$\|y\| \geq |w^*(y)| = |w^*(u+v)| = |w^*(v)| = \|v\|,$$

and we get

$$\|y\| = \|u\| = max\{\|u\|, \|v\|\}.$$

The same argument shows that if $y^* \in sp(S_2)$, then

$$\|y\| = \|v\| = max\{\|u\|, \|v\|\}.$$

□

7.3.3. THEOREM. *(Cambern) Let Y be a reflexive Banach space which has no nontrivial M summand. Then (X, Y) has the Banach-Stone property for any Banach space X.*

PROOF. Suppose that $\mathcal{H} \subset \mathfrak{F}$ is a Hamel basis for $sp(\mathfrak{F}) \subset Y^*$. Let T be an isometry from $C_0(Q, X)$ onto $C_0(K, Y)$. Because of Lemma 7.2.5, it is not difficult to see that property (P) is equivalent to the statement that for a given $t \in K$, it is the case that if R and L are any T-sets in Y and y^*, z^* are elements of $\Gamma(R) \cap ext(Y^*), \Gamma(L) \cap ext(Y^*)$, respectively, then $T^*(y^* \circ \psi_t)$ and $T^*(z^* \circ \psi_t)$ must have the same *support* in Q; i.e.,

$$T^*(y^* \circ \psi_t) = x^* \circ \psi_s \text{ and } T^*(z^* \circ \psi_t) = v^* \circ \psi_r \Longrightarrow r = s. \tag{4}$$

As a consequence of this, if we assume that (X, Y) fails to have property (P), then we may assume that for some $t \in K$, it is not true that the elements $T^*(y^* \circ \psi_t)$, for $y^* \in \mathcal{H}$, have the same support in Q. Fix $y_0^* \in \mathcal{H}$ and let

$$T^*(y_0^* \circ \psi_t) = x_0^* \circ \psi_{s_0}.$$

Let

$$S_1 = \{y^* \in \mathcal{H} : T^*(y^* \circ \psi_t) = x^* \circ \psi_{s_0}\}$$

and let $S_2 = \mathcal{H} \backslash S_1$. Since $\mathcal{H}$ is linearly independent, we have $sp(S_1) \cap sp(S_2) = \{0\}$. We will show that for any $y^* \in sp(S_1)$ with $\|y^*\| = 1$, then $\|y^* - z^*\| \geq 1$ for every $z^* \in sp(S_2)$. It will follow from this that $\overline{sp}(S_1) \cap \overline{sp}(S_2) = \{0\}$.

Thus we begin this part of the argument by letting $y^* \in sp(S_1)$ with $\|y^*\| = 1$. Then $y^* = \sum_{j=1}^{n} a_j y_j^*$, where $y_j^* \in S_1$ for each j and the a_j's are scalars. We have

$$y^* \circ \psi_t = \sum_{j=i}^{n} a_j (y_j^* \circ \psi_t)$$

as a norm 1 element of $C_0(K, Y)^*$ whose image under the isometry T^* is given by

$$T^*(y^* \circ \psi_t) = \sum_{j=1}^{n} a_j (x_j^* \circ \psi_{s_0}) = x^* \circ \psi_{s_0},$$

where $x^* = \sum_{j=1}^{n} a_j x_j^*$. Since $\|x^* \circ \psi_{s_0}\| = 1$, we also have $\|x^*\| = 1$. For any $z^* \in sp(S_2)$, $z^* = \sum_{j=1}^{m} b_j z_j^*$, where each $z_j^* \in S_2$, and

$$T^*(z^* \circ \psi_t) = \sum_{j=1}^{m} b_j(v_j^* \circ \psi_{s_j}).$$

By the definition of S_2, $s_j \neq s_0$ for each j, and we may choose a neighborhood U of s_0 which contains none of the s_j. Select $f \in C_0(Q)$ such that $f(s_0) = 1 = \|f\|$ and $f(s) = 0$ for all $s \in Q \backslash U$. Take $x \in X$ with $x^*(x) = 1 = \|x\|$ and let $F = f(\cdot)x$. Then $\|F\| = 1$ and

$$y^*(TF(t) = (x^* \circ \psi_{s_0})(F) = x^*(F(s_0)) = 1,$$

while

$$z^*(TF(t) = \sum_{j=1}^{m} b_j(v_j^* \circ \psi_{s_j})(F) = \sum b_j v_j^*(F(s_j)) = 0.$$

Therefore,

$$\|y^* - z^*\| \geq |y^*(TF(t) - z^*(TF(t)| = 1.$$

We have shown that S_1 and S_2 satisfy the hypothesis in Lemma 7.3.2, and since Y has no nontrivial M summand, by the lemma, there must exist some $z^* \in \mathfrak{F}$ such that z^* is in neither $sp(S_1)$ nor $sp(S_2)$. Since $\mathcal{H}$ is a Hamel basis for $sp(\mathfrak{F})$, there are elements $y_1^*, y_2^*, \ldots, y_n^*$ in $\mathcal{H}$ and corresponding scalars $a_1, a_2, \ldots, a_n$ such that

$$z^* = \sum_{j=1}^{n} a_j y_j^*.$$

Suppose that $T^*(z^* \circ \psi_t) = v^* \circ \psi_r$ for some $r \in Q$ and $v^* \in ext(X^*)$. Let I be the set of indices j such that $T^*(y_j^* \circ \psi_t)$ has support r. If $r = s_0$, then $\{y_j^* : j \in I\} \subset S_1$, and otherwise the set is contained in S_2. Now we also have

$$T^*(z^* \circ \psi_t) = \sum_{j=1}^{n} a_j T^*(y_j^* \circ \psi_t) = \sum_{j=1}^{n} a_j(v_j^* \circ \psi_{r_j}).$$

We next argue that the vectors v^* and v_j^* for $j \in I$ must be linearly independent. The statement is obvious if $I = \emptyset$. Otherwise, suppose there are scalars b, b_j, where $j \in I$ not all zero such that

$$bv^* + \sum_{j \in I} b_j v_j^* = 0.$$

By composing the left side with ψ_r and mapping it by $(T^*)^{-1}$, we obtain

$$b(z^* \circ \psi_t) + \sum_{j \in I} b_j(y_j^* \circ \psi_t) = 0,$$

from which it follows that

$$bz^* + \sum_{j \in I} b_j y_j^* = 0.$$

We must have $b \neq 0$ since the y_j^*'s are independent, so that if $r = s_0$, $z^* \in sp(S_1)$, and if $r \neq s_0$, then $z^* \in sp(S_2)$. Since this is impossible, we conclude that $\{v^*\} \cup \{v_j^* : j \in I\}$ is linearly independent.

By the independence just proved, there must exist some $x \in X$ such that $v^*(x) \neq 0$ but $v_j^*(x) = 0$ for each $j \in I$. Choose $f \in C_0(Q)$ so that $f(r) = 1$ and f vanishes at all supports of $T^*(y_j^* \circ \psi_t)$ where $j \notin I$, and let $F = f(\cdot)x$. Then

$$\begin{aligned} 0 \neq v^*(x) = v^*(F(r)) &= T^*(z^* \circ \psi_t)(F) \\ &= z^*(TF(t)) \\ &= \sum_{j \in I} a_j y_j^*(TF(t)) + \sum_{j \notin I} a_j y_j^*(TF(t)) \\ &= \sum_{j \in I} a_j v_j^*(F(r)) + \sum_{j \notin I} a_j v_j^*(F(s_j)) \\ &= 0 + 0. \end{aligned}$$

This contradiction finally shows the fallacy of assuming property (P) is not satisfied. The statement of the theorem now follows from Theorem 7.2.7. □

7.3.4. Corollary. *(Cambern) A reflexive Banach space has the strong Banach-Stone property if and only if it has no nontrivial M summands.*

Proof. If Y is reflexive and has no nontrivial M summand, then the above argument shows that Y has property (P), and so has the SBSP by Corollary 7.2.9. The "only if" statement follows from Example 7.3.1. □

One observation that we can easily make at this point is that although property (D) implies property (P), the converse is not true. The real space $\ell^1(3)$ does not satisfy property (D), but it has no nontrivial M summand, so it does satisfy property (P).

Part (ii) of Example 7.2.11 in Section 2 shows that every two-dimensional real space that is not isometric to $\ell^\infty(2)$ has property (D) and hence satisfies the strong Banach-Stone property. Of course, $\ell^\infty(2)$ fails the SBSP, and, as we shall now observe, it fails the weak Banach-Stone property as well. Indeed, if E is the real $\ell^\infty(n)$ and Q is compact, it is true that $C(Q, E)$ is isometric to $C(Q \times n, \mathbb{R})$, where by $Q \times n$ is meant the topological space sometimes called a *free union* or the topological sum of n copies of Q. For this, one considers n disjoint copies of Q (say by considering $Q \times \{1\}, Q \times \{2\}, \ldots, Q \times \{n\}$), the disjoint union $\cup_{j=1}^n (Q \times \{j\})$ as the underlying set, and a set U is open if and only if $U \cap (Q \times \{j\})$ is open in $Q \times \{j\}$ for each j. The operator T defined on $C(Q \times n, \mathbb{R})$ by

$$Tf(q) = (f(q,1), f(q,2), \ldots, f(q,n))$$

is easily shown to be an onto isometry. It has been shown that for each n, there are nonhomeomorphic compact Hausdorff spaces Q, K such that $Q \times n$ is homeomorphic to $K \times n$. Hence we can state the following fact.

7.3.5. THEOREM. *(Sundaresan) Given the real Banach space $E = \ell^\infty(n)$ and a positive integer $n \geq 2$, there are nonhomeomorphic compact Hausdorff spaces Q, K such that $C(Q, E)$ is isometric with $C(K, E)$. Hence E fails to have the weak Banach-Stone property.*

As a result of the above discussion, we see that the weak and strong Banach-Stone properties coincide on real two-dimensional spaces. It is natural to ask if the two properties are always the same. The answer is negative, and to see it we need only move up by one dimension. We are going to show that the only three-dimensional spaces which fail to have the weak Banach-Stone property are those which are isometric to $\ell^\infty(3)$. Then we can see that any three-dimensional space not isometric to $\ell^\infty(3)$ but which has a nontrivial M summand must necessarily satisfy the WBSP but not the SBSP. Clearly such spaces exist; for example, $\ell^2(2) \oplus_\infty \mathbb{F}$ is such a space.

7.3.6. THEOREM. *(Cambern) Suppose E is a three-dimensional Banach space. Then E fails to have the weak Banach-Stone property if, and only if, E is isometric to $\ell^\infty(3)$.*

PROOF. If E is isometric to $\ell^\infty(3)$, then E fails to have the weak Banach-Stone property by Theorem 7.3.5.

Suppose that E is not isometric to $\ell^\infty(3)$. If E has no nontrivial M-summand, then E has the SBSP and so also the WBSP. Hence suppose that $E = M \oplus_\infty N$, where M is two-dimensional and N is one-dimensional. Now M cannot have a nontrivial M summand, otherwise E will be isometric to $\ell^\infty(3)$. In this situation, M^* is isometric with N^o, the annihilator of N. If $x \in M$, and S_1 is a T-set in M containing x, then the set S given by

$$S = \{u + v : u \in S_1, v \in N, \|v\| \leq \|u\|\}$$

is a T-set in E. If $x^* \in \Gamma(S)$, then $x^* \in N^o$. Hence, if we choose $x^* \in \mathfrak{F}$ corresponding to S, the restriction of x^* to M is in M^*. In this way we can choose elements of $\mathfrak{F}$ which form a basis $\{y_1^*, y_2^*\}$ for M^*. Suppose that T is an isometry from $C_0(Q, E)$ onto $C_0(K, E)$, where Q, K are locally compact Hausdorff spaces. Since M does not have a nontrivial M summand, we can show, in exactly the same way as in the proof of Theorem 7.3.3, that for each $t \in K$, there is a unique $\varphi(t) = s \in Q$ such that $T^*(z^* \circ \psi_t) = v^* \circ \psi_s$ for each $z^* \in N^o$. This establishes the existence of a function φ from K to Q. The continuity of the function φ can be proved exactly as in the proof of Theorem 7.2.7.

Assume that $\varphi(t_1) = \varphi(t_2)$. Then for y_1^*, y_2^* as above we have extreme points $x_{ij}^* \in B(E^*)$ for $i = 1, 2;\ j = 1, 2$ such that

$$T^*(y_i^* \circ \psi_{t_j}) = x_{ij}^* \circ \psi_s$$

for each pair i, j. Since E is only three-dimensional, there must be scalars a_{ij} not all zero such that

$$\left(\sum_{i,j=1,2} a_{ij} x_{ij}^* \right) \circ \psi_s = 0.$$

If we operate on both sides of the above equation with $(T^*)^{-1}$ we obtain

$$\sum_{i,j=1,2} a_{ij} (y_i^* \circ \psi_{t_j}) = 0.$$

Assume (without loss of generality) that $a_{11} \neq 0$. Since y_1^* and y_2^* are linearly independent, there exists $u \in E$ with $y_2^*(u) = 0 \neq y_1^*(u)$. We may choose $G \in C_0(K, E)$ such that $G(t_1) = u$ and $G(t_2) = 0$. Then

$$0 = \sum_{i,j=1,2} a_{ij} (y_i^* \circ \psi_{t_j})(G) = a_{11} y_1^*(u) \neq 0.$$

This contradiction shows that φ must be one-to-one.

Next suppose that there is some $s \in Q$ such that $s \notin \varphi(K)$. Again, using the elements y_1^*, y_2^* of $\mathfrak{F}$ that are a basis for $N^o = M^*$, and applying the argument of Theorem 7.3.3 to T^{-1}, we have linearly independent elements v_1^*, v_2^* of the unit ball E^* and some $r \in K$ such that

$$(T^*)^{-1}(y_j^* \circ \psi_s) = v_j^* \circ \psi_r$$

for $j = 1, 2$. Now we must have $\varphi(r) = q \neq s$. The elements $y_1^*, y_2^*, v_1^*, v_2^*$ are linearly dependent, and we can give an argument exactly as in the previous paragraph that leads to a contradiction. Hence φ is onto.

Finally, we can show that φ^{-1} is continuous using the standard argument as in Theorem 7.2.7. This completes the proof that Q and K are homeomorphic. □

Let us end this section with one more positive result. We are going to show that if Y contains no copy of real $\ell^\infty(2)$, then Y is universal for the Banach-Stone property. The key idea in our proof is the fact that Y does not contain a copy of real $\ell^\infty(2)$ if and only if at least one of the norms $\|x \pm y\| < 2$ whenever $\|x\| = \|y\| = 1$.

7.3.7. THEOREM. *(Jeang and Wong) If Y is a Banach space which contains no copy of real $\ell^\infty(2)$, then (X, Y) has property (P) for every Banach space X. Furthermore,*

(i) *(X, Y) has the Banach-Stone property for each X;*
(ii) *if both X and Y contain no copy of $\ell^\infty(2)$, then (X, Y) has the strong Banach-Stone property; and*
(iii) *Y has the strong Banach-Stone property.*

PROOF. Suppose X is a Banach space such that (X, Y) fails property (P). Then there are an isometry T from $C_0(Q, X)$ onto $C_0(K, Y)$, and T-sets $(R_1, t), (R_2, t)$ such that $T^{-1}(R_1, t)$ is based at s, and $T^{-1}(R_2, t)$ is based at r with $r \neq s$. This means that there are T-sets S_1, S_2 in X such that $T(S_1, s) = (R_1, t)$ and $T(S_2, r) = (R_2, t)$. Moreover, by Lemma 7.2.5, given $y^* \in ext(Y^*) \cap \Gamma(R_1)$ and $z^* \in ext(X^*) \cap \Gamma(R_2)$, there are x^*, v^* in $ext(X^*) \cap \Gamma(S_1)$ and $ext(X^*) \cap \Gamma(S_2)$, respectively, such that

$$T^*(y^* \circ \psi_t) = x^* \circ \psi_s \text{ and } T^*(z^* \circ \psi_t) = v^* \circ \psi_r. \tag{5}$$

We can construct functions F_1, F_2 in $C_0(Q, X)$ which have support on disjoint neighborhoods of s,r, respectively, such that $1 = \|F_1\| = \|F_1(s)\| = \|F_1(r)\| = \|F_2\|$. We can also construct these functions in such a way that $x^*(F_1(s)) = v^*(F_2(r)) = 1$. It follows from (5) that

$$y^*(TF_1(t)) = 1, \text{ and } z^*(TF_2(t)) = 1.$$

Hence, $\|TF_1(t)\| = \|TF_2(t)\| = 1$.

The properties of F_1 and F_2 imply that $\|F_1 \pm F_2\| = 1$ and since T is an isometry we must conclude that

$$\|TF_1(t) \pm TF_2(t)\| \leq 1.$$

However,

$$\begin{aligned} 2 = 2\|TF_1(t)\| &= \|T(F_1 + F_2)(t) + T(F_1 - F_2)(t)\| \\ &\leq \|T(F_1 + F_2)(t)\| + \|T(F_1 - F_2)(t)\| \\ &\leq 2 \end{aligned}$$

from which we conclude that $\|T(F_1 \pm F_2)(t)\| = 1$. Since Y contains no copy of $\ell^\infty(2)$, we must have

$$\|T(F_1 + F_2)(t) \pm T(F_1 - F_2)(t)\| < 2$$

for one of the norms. This will contradict either $\|TF_1(t)\| = 1$ or $\|TF_2(t)\| = 1$.

Thus we must have (X, Y) has property (P). The other properties mentioned follow directly from this.

□

We note that if Y has no copy of $\ell^\infty(2)$, then it can have no M summand, and so if Y is also reflexive, this last result follows from Theorem 7.3.3. On the other hand, a space can have a copy of $\ell^\infty(2)$ and still have no M summand. For example, the space $Y = \mathbb{R} \oplus_2 (\mathbb{R} \oplus_\infty \mathbb{R})$ contains a copy of $\ell^\infty(2)$ but still has the strong Banach-Stone property, since it is reflexive and contains no M summand.

7.4. Centralizers, Function Modules, and Behrends' Theorem

We have seen so far that a Banach space Y has the strong Banach-Stone property if and only if it satisfies property (P), and that Y satisfies property (P) if it is strictly convex or is reflexive and contains no nontrivial M summand. In this section we want to consider a condition on Y that will imply that it satisfies property (P), and which is more general than any of the conditions we have identified previously. It involves the notion of the *centralizer* of a Banach space. The ideas to be discussed are a part of what has come to be called M structure theory of Banach spaces. We will also take the opportunity to discuss a class of spaces, called *Banach function modules*, which are more general than, but certainly inspired by, the spaces $C_0(Q, X)$ we have been studying.

7.4.1. DEFINITION. *Let T be a bounded linear operator on a Banach space X.*

(i) *The operator T is called a* multiplier *of X if every element of $ext(X^*)$ is an eigenvector for T^*. Hence, for each $x^* \in ext(X^*)$ we have a scalar $a_T(x^*)$ such that*
$$T^* x^* = a_T(x^*) x^*.$$
(ii) *The operator T is said to be M* bounded *if there is a $\lambda > 0$ such that, for every $x \in X$, Tx is contained in every ball which contains $\{\mu x : \mu \in \mathbb{F},\ |\mu| \le \lambda\}$.*
(iii) *For a multiplier T on X, we say that a multiplier S on X is an* adjoint *for T if $a_S = \overline{a_T}$. If T has an adjoint, we will denote it by T^a.*
(iv) *The* centralizer *of X, written as $Z(X)$, is the set of all multipliers for which an adjoint exists. (Note that in case $\mathbb{F} = \mathbb{R}$, the centralizer just consists of the multipliers.)*

We note here for future reference that it can be shown that an operator T is a multiplier if and only if it is M bounded.

7.4.2. DEFINITION. *A* Banach function module*, or* function module *is a triple $(K, (Y_t)_{t\in K}, Y)$, where K is a nonempty compact Hausdorff space (the* base space*), $(Y_t)_{t\in K}$ a family of Banach spaces (the* component spaces*), and Y a closed subspace of $\Pi^{\infty}_{t\in K} Y_t$ such that the following conditions are satisfied:*

(i) *$hy \in Y$ for $y \in Y$ and $h \in C(K)$ ($(hy)(t) = h(t)y(t)$);*
(ii) *$t \to \|y(t)\|$ is an upper semicontinuous function for every $y \in Y$;*
(iii) *$Y_t = \{y(t) : y \in Y\}$ for every $t \in K$;*
(iv) *$\{t \in K : Y_t \ne \{0\}\}^- = K$.*

Note: the space $\Pi^{\infty}_{t\in K} Y_t$ denotes the functions y in the product space for which
$$\|y\|_\infty = \sup\{\|y(t)\| : t \in K\} < \infty.$$

A natural example of a function module is a space $C(K, E)$ where K is compact and we take $Y_t = E$ for each $t \in K$. If K is locally compact, we replace K by its Stone-Cêch compactification βK and take $Y_t = E$ if $t \in K$ and $Y_t = \{0\}$ if $t \in \beta K \backslash K$.

A property of function modules that will be of importance to us is that for any $t \in K$, there is some element F of the module which attains its norm on t. Even a bit more can be said.

7.4.3. LEMMA. *If $(K, (Y_t)_{t\in K}, Y)$ is a function module and $t \in K$, $u \in Y_t$ are given, there exists $F \in Y$ such that $F(t) = u$ and $\|F\| = \|F(t)\| = \|u\|$. Furthermore, if U is a given neighborhood of t, F may be chosen so that $F(r) = 0$ for $r \in K \backslash U$.*

PROOF. If u, t are given, suppose $G \in Y$ is such that $G(t) = u$, and let $U = U_0$ be an open neighborhood of t. Since the map $t \to \|G(t)\|$ is uppersemicontinuous, for each positive integer n there is an open neighborhood $U_n \subset U_{n-1}$ such that

$$\sup\{\|G(r)\| : r \in U_n\} < \|G(t)\| \left(\sum_{j=1}^{n} 2^{-j}\right)^{-1}.$$

For each n we choose $h_n \in C(K)$ such that the range of h_n is in $[0, 1]$, $h_n(t) = 1$, and $h_n(r) = 0$ for all $r \in K \backslash U_n$. Then the function $h = \sum_{j=1}^{\infty} 2^{-j} h_j$ is continuous on K, has value 1 at t, and vanishes outside U. The function $F = hG$ is in Y by the definition of a function module. If $s \in U_n$ for every n, we have

$$\|F(s)\| \leq \left(\sum_{j=1}^{\infty} 2^{-j}\right) \|G(t)\| \left(\sum_{j=1}^{n} 2^{-j}\right)^{-1}$$

for all n so that $\|F(s)\| \leq \|G(t)\| = \|F(t)\|$. If $s \in U_n \backslash U_{n+1}$, then $h_j(s) = 0$ for $j \geq n+1$ and

$$\|F(s)\| \leq \left(\sum_{j=1}^{n} 2^{-j}\right) \|G(t)\| \left(\sum_{j=1}^{n} 2^{-j}\right)^{-1} = \|F(t)\|.$$

If $s \notin U_1$, then $h = 0$. Thus, $\|F\| \leq \|F(t)\|$, and F has the desired properties. □

It is true that any Banach space X can be thought of as a function module. A *function module representation* $[\rho, (K, (Y_t)_{t\in K}, Y)]$ of X is a function module $(K, (Y_t)_{t\in K}, Y)$ together with an isometric isomorphism $\rho : X \to Y$. If for $h \in C(K)$, we let M_h be the multiplication operator on Y defined by $M_h F(t) = h(t)F(t)$, then $Z_\rho(X) = \{\rho^{-1} M_h \rho : h \in C(K)\}$ is contained in $Z(X)$. Given any Banach space X, there is a function module representation of X with the property that $Z_\rho(X) = Z(X)$. We are not, however, going to make direct use of this idea here.

It is easy to see that for $h \in C(K)$, M_h is a multiplier with adjoint $M_{\bar{h}}$ and so lies in the centralizer of the function module Y. In fact, the centralizer of $C(K)$ itself consists exactly of the set $\{M_h : h \in C(K)\}$. In the locally compact case, $Z(C_0(K))$ consists of multiplications by elements of $C^b(K)$, the bounded continuous functions on K. We are going to see that for purposes of obtaining Banach-Stone theorems for vector-valued functions, good things happen when the centralizers consist of exactly these multipliers.

As we have previously seen, the key to obtaining theorems of the Banach-Stone type is in constructing a well-defined function φ from the locally compact space K to the locally compact space Q. In the previous sections, the notion of T-set was fundamental to a successful outcome. Now we intend to use the characterization of extreme points of $C_0(K,E)^*$ and the fact that the conjugate T^* of an isometry T must map extreme points to extreme points. In Chapter 2, we spent considerable space in showing that the extreme points of $C_0(K,E)$ were of the form $y^* \circ \psi_t$, where $y^* \in ext(E^*)$ and $t \in K$. We have used this characterization in Sections 2 and 3 of the present chapter. Thus, if T is an isometry from $C_0(Q,X)$ onto $C_0(K,Y)$, then for any $t \in K$ and $y^* \in ext(Y^*)$ we have

$$T^*(y^* \circ \psi_t) = x^* \circ \psi_s, \tag{6}$$

where $s \in Q$ and $x^* \in ext(X^*)$. We let $\varphi(t) = s$, and the problem is to show that φ is well defined. We know that conditions are needed on Y before this can be established. The conditions we will use have to do with the size of the centralizer of the space Y. We are going to begin by describing the centralizer of a Banach space Y which is itself a function module. For that we will need a characterization of the extreme points of the unit ball of the conjugate of Y. Fortunately, such a characterization is available to us, and we are going to borrow it without giving a proof.

7.4.4. Lemma. *(Behrends) If $(K,(Y_t)_{t\in K},Y)$ is a function module, then γ is an extreme point of $B(Y^*)$ if and only if there are $t \in K$ and $y^* \in ext(Y_t^*)$ such that $\gamma = y^* \circ \psi_t$.*

For a proof of this result, see [**27**, p. 80]. It clearly includes the characterization of extreme points for the unit ball of $C_0(K,E)^*$ alluded to above, and which we obtained by other methods in Chapter 2.

We will say that a Banach space E has *trivial centralizer* if the dimension of $Z(E)$ is equal to 1; that is, if the only elements in the centralizer are scalar multiples of the identity operator I. Obviously this is true if E is itself the scalar field.

7.4.5. Lemma. *Let $(K,(X_t)_{t\in K},X)$ be a function module with the property that $Z(X_t)$ is trivial for each $t \in K$. Then*

$$Z(X) = \{M_h : h \text{ is bounded on } K \text{ and } hF \in X \text{ for all } F \in X\}.$$

Proof. If h is a bounded scalar-valued function on K such that $hF \in X$ for all $F \in X$, then it is straightforward to show M_h is in the centralizer of

X. Suppose, on the other hand, that $W \in Z(X)$. Given $t \in K_0 = \{t \in K : X_t \neq \{0\}\}$ and $x^* \in ext(X_t^*)$ we have a scalar $a_W(x^*, t)$ such that

$$W^*(x^* \circ \psi_t) = a_W(x^*, t)(x^* \circ \psi_t).$$

Note that

$$(7) \quad |a_W(x^*, t)| = \|a_W(x^*, t)(x^* \circ \psi_t)\| = \|W^*(x^* \circ \psi_t)\| \leq \|W^*\| = \|W\|.$$

Let us define $P(t)$ on X_t by

$$P(t)u = WF(t), \text{ where } F \in X \text{ with } F(t) = u.$$

Now $P(t)$ is well defined, for if $F(t) = H(t)$, then

$$\begin{aligned} x^*(WF(t)) = a_W(x^*, t)x^*(F(t)) &= a_W(x^*, t)x^*(H(t)) \\ &= a_W(x^*, t)(x^* \circ \psi_t)(H) \\ &= W^*(x^* \circ \psi_t)(H) \\ &= x^*(WH(t)) \end{aligned}$$

for all $x^* \in ext(X_t^*)$. Hence, $WF(t) = WH(t)$. Moreover, $P(t)$ is bounded, for if $u \in X_t$, and $F(t) = u$, by Lemma 7.2.4 there exists $x^* \in ext(X_t^*)$ such that $x^*(WF(t)) = \|WF(t)\|$. Thus, from (7) we obtain

$$\begin{aligned} \|P(t)u\| = \|WF(t)\| &= W^*(x^* \circ \psi_t)(F) \\ &= |a_W(x^*, t)(x^* \circ \psi_t)(F)| \\ &= |a_W(x^*, t)x^*(F(t))| \\ &= |a_W(x^*, t)|\|u\| \leq \|W\|\|u\|. \end{aligned}$$

Similar manipulations show that

$$P^*(t)x^* = a_W(x^*, t)x^*,$$

and $P(t)$ is a multiplier. Since it will have an adjoint obtained from the adjoint W^a of W, we conclude that $P(t) \in Z(X_t^*)$. Consequently, there is a scalar $h(t)$ such that $P(t) = h(t)I$ and we have

$$WF(t) = h(t)F(t)$$

for all $t \in K_0$, where $h(t) = a_W(x^*, t)$ for all x^*. That h is bounded on K_0 follows from (7). It does not matter what values h is given outside K_0; they can be any finite constant. □

7.4.6. Corollary. *Let K be a locally compact Hausdorff space and E a Banach space with trivial centralizer. Then $Z(C_0(K, E)) = \{M_h : h \in C^b(K)\}$.*

Proof. Since $X = C_0(K, E)$ is a function module, the form of elements in the centralizer comes immediately from Lemma 7.4.5. (Recall that the compact set in this case is βK and the set K_0 of the lemma is identified here with K.) It is only necessary to show that for $M_h \in Z(X)$, h is continuous at each $t \in K$. Given $t_0 \in K$, there exist a compact neighborhood U of t_0, $u \in E$ with $\|u\| = 1$ and $F \in C_0(K, E)$ such that $F(t) = u$ for all $t \in U$.

Since hF is continuous at t_0, given $\epsilon > 0$, there is a neighborhood U_0 of t_0 and contained in U such that $\|hF(t) - hF(t_0\| < \epsilon$ for all $t \in U_0$. Therefore, for such t we have

$$|h(t) - h(t_0)| = \|h(t)u - h(t_0)u\| = \|hF(t) - hF(t_0)\| < \epsilon.$$

□

We are now ready to characterize isometries from one function module onto another. As we mentioned earlier, the conjugate of an isometry must map extreme points to extreme points. In order to emphasize the fact that it is this property more than the isometric property that is needed, we introduce a class of operators which includes the isometries as a subclass.

7.4.7. Definition. *Given Banach spaces X, Y, an operator $T \in \mathcal{L}(X, Y)$ is said to be* nice *if $T^*(ext(Y^*)) \subseteq ext(X^*)$.*

Although every isometry is nice, not every nice operator, nor even a nice isomorphism, is necessarily an isometry. As a simple example of this, let ν be a non-strictly convex norm on $\mathbb{R}^2$ whose dual ball has extreme points as a proper subset of the unit circle. Let E_1 be $\mathbb{R}^2$ with Euclidean norm and let E_2 be $\mathbb{R}^2$ with the norm ν. Let $V(1) = V(2)$ be a nice linear operator from E_1 to E_2 (e.g., the identity), and let φ be a permutation of $\{1, 2\}$. Then $TF(t) = V(t)F(\varphi(t))$ defines a nice isomorphism from $X = C(\{1, 2\}, E_1)$ onto $Y = C(\{1, 2\}, E_2)$ which is not an isometry and for which $T^*(ext(Y^*)) \neq ext(X^*)$. Of course, if T^* is an isometry, and T is an isomorphism, then T will be an isometry.

Note that a nice operator is necessarily a contraction. For, if $x \in X$ and if $y^* \in ext(Y^*)$ so that $|y^*(Tx)| = \|Tx\|$, then

$$\|Tx\| = |y^*(Tx)| = |T^*y^*(x)| \leq \|x\|.$$

7.4.8. Theorem. *Suppose T is a nice isomorphism from the function module $X = (Q, (X_s)_{s\in Q}, X)$ onto the function module $(K, (Y_t)_{t\in K}, Y)$ where $Z(Y_t)$ is trivial for each $t \in K$ such that $Y_t \neq \{0\}$. Then there is a function φ from $K_0 = \{t \in K : Y_t \neq 0\}$ onto a dense subset of Q and a function $t \to V(t)$ from K_0 into the family of nice operators from $X_{\varphi(t)}$ to Y_t such that*

$$TF(t) = V(t)F(\varphi(t)) \tag{8}$$

for all $t \in K_0$ and $F \in X$.

Proof. Let h be a continuous function on Q. We will first show that TM_hT^{-1} is in the centralizer of Y. If $t \in K_0$, $y^* \in ext(Y_t^*)$, then, since T is nice, there are $s \in Q$ and $x^* \in ext(X_s^*)$ such that $T^*(y^* \circ \psi_t) = x^* \circ \psi_s$.

Hence, for $G = TF \in Y$, we have

$$\begin{aligned}(TM_hT^{-1})^*(y^* \circ \psi_t)(G) &= (T^{-1})^* M_h^* T^*(y^* \circ \psi_t)(G) \\ &= T^*(y^* \circ \psi_t)(M_hF) \\ &= (x^* \circ \psi_s)(M_hF) \\ &= x^*(h(s)F(s)) \\ &= h(s)x^*(F(s)) \\ &= h(s)T^*(y^* \circ \psi_t)(F) \\ &= h(s)(y^* \circ \psi_t)(G).\end{aligned}$$

This shows that each extreme point of $S(Y^*)$ is an eigenvector and so TM_hT^{-1} is a multiplier. Its adjoint will be given by $TM_{\bar{h}}T^{-1}$ and so TM_hT^{-1} is in the centralizer. By Lemma 7.4.5, there is a bounded function $\tilde{h}$ on K such that $TM_hT^{-1} = M_{\tilde{h}}$. This implies that

$$TM_h = M_{\tilde{h}}T.$$

If we proceed as above it is easy to show that

$$\begin{aligned}h(s)x^*(F(s)) &= (TM_h)^*(y^* \circ \psi_t)(F) \\ &= (M_{\tilde{h}})^*(y^* \circ \psi_t)(TF) \\ &= \tilde{h}(t)x^*(F(s))\end{aligned}$$

for any $F \in X$. Since $x^* \circ \psi_s$ is an extreme point we can choose F so that $x^*(F(s)) \neq 0$. Thus we must conclude that

$$h(s) = \tilde{h}(t).$$

Now, if we have $y^*, z^* \in ext(Y_t^*)$ and

$$T^*(y^* \circ \psi_t) = x^* \circ \psi_s \text{ and } T^*(z^* \circ \psi_t) = w^* \circ \psi_r, \tag{9}$$

it follows that

$$h(s) = \tilde{h}(t) = h(r).$$

Since there are enough continuous functions on Q to separate points of Q, we must have $r = s$. We define the function φ on K_0 by $\varphi(t) = s$ according to the pairing determined by T^* as in (6). Then φ is well defined, since (9) implies $r = s$.

Next we define, for each $t \in K_0$, an operator $V(t)$ on $X_s = X_{\varphi(t)}$ by

$$V(t)u = TF(t)$$

where $F \in X$ has the property that $F(\varphi(t)) = u$. To see that $V(t)$ is well defined, suppose that $s = \varphi(t)$ and $F(s) = H(s)$. Then for any extreme point y^* for $B(Y_t^*)$, and $T^*(y^* \circ \psi_t) = x^* \circ \psi_s$, we have

$$y^*(TF(t)) = x^*(F(s)) = x^*(H(s)) = y^*(TH(t)),$$

from which it follows that $TF(t) = TH(t)$. To see that $V(t)$ is bounded, let $u \in X_{\varphi(t)}$ be given and $F \in X$ so that $F(\varphi(t)) = u$. There exists $y^* \in ext(Y_t^*)$ with $\|V(t)u\| = \|TF(t)\| = y^*(TF(t))$. Hence,

$$\|V(t)u\| = y^*(TF(t)) = T^*(y^* \circ \psi_t)(F) = x^*(F(\varphi(t)) \leq \|u\|.$$

It is straightforward to show that $V(t)^*$ maps extreme points of Y_t^* to extreme points of $X_{\varphi(t)}^*$. Hence we have established (8), since by its definition, $V(t)F(\varphi(t)) = TF(t)$.

To complete the proof, we show that $\varphi(K_0)$ is dense in Q. Suppose s_0 is such that $X_{s_0} \neq 0$ and $s_0 \in Q \backslash \varphi(K_0)^-$. There exists some $F \in X$ such that $F(s_0) \neq 0$ and $h \in C(Q)$ with $h(s_0) = 1$ and $h(s) = 0$ for all $s \in \varphi(K_0)^-$. If $H = hF$, then H is a nonzero element of X but

$$TH(t) = V(t)H(\varphi(t)) = 0$$

for all $t \in K_0$. This contradicts the fact that T is injective, and from part (iv) of Definition 7.4.2, we conclude that $\varphi(K_0)$ is dense. □

If we assume that the nice operator is actually an isometry, we can prove a tiny bit more.

7.4.9. Corollary.

(i) *Suppose that T is an isometry defined from the function module $X = (Q, (X_s)_{s \in Q}, X)$ onto the function module $Y = (K, (Y_t)_{t \in K}, Y)$ where $Z(Y_t)$ is trivial for each $t \in K$ such that $Y_t \neq \{0\}$. Then there is a function φ from $K_0 = \{t \in K : Y_t \neq \{0\}\}$ onto $Q_0 = \{s \in Q : X_s \neq \{0\}\}$ and a function $t \to V(t)$ from K_0 into the family of nice operators from $X_{\varphi(t)}$ to Y_t such that*

$$TF(t) = V(t)F(\varphi(t)) \tag{10}$$

for all $t \in K_0$ and $F \in X$.

(ii) *If, in addition, we assume that $Z(X_s)$ is trivial for each $s \in Q_0$, then $V(t)$ is an isometry for each $t \in K_0$.*

Proof. (i) Since an isometry is nice, we apply Theorem 7.4.8 to get the function φ and the operator $V(t)$ to satisfy (10). Note that the function φ is necessarily onto Q_0 in this case, because for any $s \in Q_0$, and $x^* \in ext(X_s^*)$, we have $(T^*)^{-1}(x^* \circ \psi_s) = y^* \circ \psi_t$ for some $t \in K$ and $y^* \in ext(Y_t^*)$. Therefore, $\varphi(t) = s$.

(ii) In this case, we can apply the first part to T^{-1} so that the function φ is necessarily one-to-one. Let $s = \varphi(t)$, $u \in X_s$ and suppose $x^* \in ext(X_s^*)$ so that $x^*(u) = \|u\|$. Let $F \in X$ with the property that $F(s) = u$ and

$\|F\| = \|F(s)\|$. We must have

$$\begin{aligned}\|u\| = x^*(u) = x^*(F(s)) &= (x^* \circ \psi_s)(F) \\ &= T^*(y^* \circ \psi_t)(TF) \\ &= y^*(TF(t)) \\ &\leq \|TF(t)\| \leq \|TF\| \\ &= \|F\| = \|u\|.\end{aligned}$$

Hence $\|u\| = \|TF(t)\| = \|V(t)u\|$, and $V(t)$ is an isometry. □

7.4.10. THEOREM.

(i) *Let T be a nice isomorphism from $C_0(Q, X)$ onto $C_0(K, Y)$ and suppose that Y has a trivial centralizer. Then there exists a continuous function φ from K onto a dense subset of Q and a continuous function $t \to V(t)$ from K into the collection of nice operators contained in $\mathcal{L}(E_1, E_2)$ (given the strong operator topology) such that*

$$TF(t) = V(t)F(\varphi(t))$$

for all $t \in K$ and $F \in C_0(Q, X)$.

(ii) *If T is an isometry, then the continuous function φ maps K onto Q.*

(iii) *If T is an isometry, and both X and Y have trivial centralizers, then φ is a homeomorphism from K onto Q, and each $V(t)$ is an isometry.*

PROOF. (i) The canonical form comes immediately from Theorem 7.4.8, since the spaces involved are function modules. In this identification, recall that K plays the role of K_0 in the statement of the theorem. The continuity of φ can be proved exactly as in the proof of Theorem 7.2.7, exploiting the continuity of F and TF which we have in the current setting. The continuity of $t :\to V(t)$ is also proved exactly as before, in Theorem 7.2.7.

(ii) and (iii) These are immediate from Corollary 7.4.9. □

7.4.11. COROLLARY. *(Behrends) If X, Y are Banach spaces and Y has a trivial centralizer, then (X, Y) has the Banach-Stone property. Furthermore, if X, Y both have trivial centralizers, then (X, Y) has the strong Banach-Stone property. Finally, if Y has a trivial centralizer, then Y has the strong Banach-Stone property.*

PROOF. The first statement is essentially the content of Theorem 7.4.10(i) and (ii). For the second statement, we apply Theorem 7.4.10(iii). The last statement of the corollary is obvious from the second statement. □

The reader may have noted that the argument given in the first part of the proof of Theorem 7.4.8 shows that if Y has a trivial centralizer, then (X, Y) satisfies property (P). Hence the previous corollary also follows from Theorem 7.2.7, Corollary 7.2.8, and Corollary 7.2.9. Let us state this fact formally.

7.4.12. THEOREM. *If Y has a trivial centralizer, then (X, Y) has property (P) for every Banach space X.*

PROOF. The argument given in the first part of the proof of Theorem 7.4.8 shows that if T is an isometry from $C_0(Q, X)$ onto $C_0(K, Y)$, then (4) is satisfied, and from the remark given there at the beginning of the proof of Theorem 7.3.3, we conclude that (X, Y) has property (P). □

It is natural now to ask whether there are spaces which have trivial centralizers. Obviously this is true for one-dimensional spaces. On the other hand, for $X = \ell^\infty(2) = C(\{1, 2\}, \mathbb{R})$, $Z(X)$ has dimension 2 since multiplication by the functions

$$h_j(t) = \begin{cases} 1, & \text{if } t = j; \\ 0, & \text{otherwise.} \end{cases}$$

gives two linearly independent multipliers. Happily, there are many spaces with trivial centralizers, enough, in fact, that Corollary 7.4.11 actually includes the theorems proved in Sections 2 and 3. We collect information on this in the theorem below. The proof will be mostly a sketch, since full details would take us more deeply into M structure theory than we want to go. Once again, we must first introduce some terminology.

7.4.13. DEFINITION.

(i) *A closed subspace J of a Banach space X is said to be an L* summand *(M* summand*) if there is a closed subspace $J^\perp$ such that X is the algebraic direct sum of J and $J^\perp$ (i.e., $X = J \oplus J^\perp$), and*

$$\|x + y\| = \|x\| + \|y\| \quad (\|x + y\| = max\{\|x\|, \|y\|\})$$

for $x \in J$, $y \in J^\perp$. The associated projections on an L summand and an M summand are called L projections *and M* projections*, respectively.*

(ii) *A closed subspace J of a Banach space X is said to be an M* ideal *if J^o, the annihilator of J in X^*, is an L summand in X^*.*

(iii) *A Banach space X is* smooth *if at every point on the surface of the unit ball there exists only one supporting hyperplane.*

7.4.14. THEOREM. *(Behrends) Let X be a nonzero Banach space. Each of the following conditions implies that $Z(X)$ is one-dimensional.*

(i) *X is smooth;*
(ii) *X is strictly convex;*
(iii) *X has no nontrivial M ideal;*
(iv) *X is reflexive and all M summands of X are trivial;*
(v) *X has property (D);*
(vi) *$Z(X_\mathbb{R})$ is one-dimensional, where $X_\mathbb{R}$ denotes X regarded as a real space;*
(vii) *X is not isometric with real $\ell^\infty(2)$, and there exists a nontrivial L projection on X;*

(viii) *X is not isometric with real $\ell^\infty(2)$, and X^* contains a nontrivial M ideal.*

Proof. We begin with the observation that if $(Q, (X_s)_{s\in Q}, X)$ is a function module and Q contains more than one point, then X is neither smooth nor strictly convex, and X has a nontrivial M ideal. For, if Q contains more than two points, we can find norm 1 elements $F, G \in X$ such that $F(s) = 0$ or $G(s) = 0$ for all $s \in Q$. Then the span of F, G is isometric to $\ell^\infty(2)$, so X cannot be strictly convex or smooth. Furthermore, for any $s \in Q$, the set $\{F : F(s) = 0\}$ is a nontrivial M ideal in X.

For any Banach space X, the centralizer $Z(X)$ is a commutative B^*-algebra. Hence, by the Gelfand-Naimark theorem, there is an isometric isomorphism from $Z(X)$ onto a space $C(K_X)$, where K_X is a compact Hausdorff space, uniquely determined by the classical Banach-Stone theorem. It is known that there is a function module representation $[\rho, (K_X, (Y_t)_{t\in K_X}, Y)]$ of X such that $Z_\rho(X) = Z(X)$. Now, from our observation above, if either (i), (ii), or (iii) is satisfied for X, then K_X must be a singleton, from which we conclude that $C(K_X)$ and $Z(X)$ are one-dimensional.

If X is reflexive, then every M ideal is an M summand, so in this case, (iv) implies (iii), and $Z(X)$ is one-dimensional.

It can be shown that, just as in the case of $C_0(Q, X)$, a general T-set in a function module $(Q, (X_s)_{s\in Q}, X)$ is of the form

$$(S, s) = \{F \in X : \|F\| = \|F(s)\|, F(s) \in S\},$$

where S is a T-set in X_s. It follows that if every two T-sets in X are discrepant, then Q must consist of one point. Using the representation idea above, if X is a Banach space which satisfies property (D), then K_X must be a singleton and $Z(X)$ is one-dimensional.

For (vi), it is enough to observe that for a complex space X, $Z(X) = Z(X_\mathbb{R}) + iZ(X_\mathbb{R})$.

Finally, it is the case that if X is a Banach space which is not isometrically isomorphic to real $\ell^\infty(2)$, then either all L summands of X or all M summands of X are trivial. Thus if (vii) holds, X has no nontrivial M ideal. Otherwise, X^* would have both nontrivial L summands and M summands. Similarly, if (viii) is satisfied, then X^{**} contains a nontrivial L summand, and so if X also contains a nontrivial M ideal, X^{**} would necessarily also contain a nontrivial M summand which is a contradiction. Therefore, in either (vii) or (viii), we conclude that X has no nontrivial M ideal, and by (iii), $Z(X)$ is one-dimensional. □

Note that Theorem 7.2.13, Corollary 7.2.14, Corollary 7.2.15, Theorem 7.2.16, Theorem 7.2.17, and Theorem 7.3.3 and its corollary all follow from Corollary 7.4.11 and the appropriate part of Theorem 7.4.14. In addition, there are spaces that were not covered by any of those results. For example, ℓ^1 is a nonreflexive, non-strictly convex space which fails property (D), but which has a one-dimensional centralizer by Theorem 7.4.14 (vii).

7.5. The Nonsurjective Vector-Valued Case

In the previous sections, the isometries under discussion were always assumed to be surjective. In parallel with our approach in Section 3 of Chapter 2, we now wish to consider isometries whose range is a subspace of a continuous, vector-valued function space. As we shall see, certain difficulties are inherent in this setting, and we intend to investigate several approaches to overcoming them. We begin with a theorem that will reveal the major change in the description of into isometries from what we have seen in the previous work of this chapter. As usual, it will be helpful to introduce some new notation. We assume now that T is an isometry from $C_0(Q, X)$ onto a subspace N of $C_0(K, Y)$, where, as before, Q, K are locally compact Hausdorff spaces and X, Y are Banach spaces. Given $x \in S(X)$ and $s \in Q$, let

$$\mathfrak{F}(x, s) = \{F \in C_0(Q, X) : F(s) = \|F\|x\}, \text{ and}$$

$$\mathcal{B}(x, s) = \{t \in K : \|TF(t)\| = \|F\| \text{ for } F \in \mathfrak{F}(x, s)\}.$$

Also, we let

$$\mathcal{B}(s) = \bigcup_{x \in S(X)} \mathcal{B}(x, s) \text{ and } \mathcal{B}(T) = \bigcup_{s \in Q} \mathcal{B}(s).$$

7.5.1. Lemma. *For each pair x, s, with $x \in S(X)$ and $s \in Q$, $\mathcal{B}(x, s) \neq \emptyset$.*

Proof. Given $x \in S(X)$, choose $x^* \in ext(X^*)$ such that $x^*(x) = 1$. Then $x^* \circ \psi_s$ is an extreme point for the dual of $C_0(Q, X)$, so that $(T^{-1})^*(x^* \circ \psi_s) = y^* \circ \psi_t$ for some $t \in K$ and $y^* \in ext(Y^*)$. Given $F \in \mathfrak{F}(x, s)$, we have

$$y^*(TF(t)) = (x^* \circ \psi_s)(F) = x^*(F(s)) = \|F\|,$$

and $t \in \mathcal{B}(x, s)$. □

The next lemma is the crucial one, and should be compared to Lemma 7.2.6 in Section 2.

7.5.2. Lemma. *If $t \in \mathcal{B}(s)$, and $F(s) = 0$ for some $F \in C_0(Q, X)$, then $TF(t) = 0$.*

Proof. Since $t \in \mathcal{B}(s)$, there exists $x \in S(X)$ such that $t \in \mathcal{B}(x, s)$. Assume that F is an element of $C_0(Q, X)$ which vanishes in a neighborhood U of s, and that $\|F\| < 1$. By the complete regularity of Q, we can choose $f \in C_0(Q)$ with $f(s) = \|f\| = 1$ and such that $f(r) = 0$ for $r \in Q \backslash U$. Let $F_1 = f(\cdot)x$, $F_2 = F + F_1$, and $F_3 = \frac{1}{2}(F_1 + F_2)$. It is easy to show, since F and F_1 have disjoint supports, that each of the functions F_i has norm 1, and furthermore, $F_j(s) = f(s)x = \|F_j\|x$ for each $j = 1, 2, 3$. Since $t \in \mathcal{B}(x, s)$, we must have that $\|TF_j(t)\| = 1$ for each i. Since the $TF_j(t)$ are all elements of the unit sphere of the strictly convex space Y, and $TF_3(t) = \frac{1}{2}TF_1 + \frac{1}{2}TF_2(t)$, we must have $TF_3(t) = TF_2(t) = TF_1(t)$. Hence, $TF_1(t) = TF_1(t) + TF(t)$ and we conclude that $TF(t) = 0$.

Now let F be an arbitrary element of $C_0(Q, X)$ with $F(s) = 0$. Given $\epsilon > 0$, there exist a neighborhood U of s and a compact set D containing U so that $\|F(r)\| < \epsilon/2$ for $r \in U \bigcup (Q \backslash D)$. Furthermore, we may choose a continuous function g on Q with range in $[0, 1]$ and which has the value 0 on a closed neighborhood V of s contained in U and also has the value 1 on $D \backslash U$. Then $gF \in C_0(Q, X)$ and vanishes on a neighborhood of s, so that $T(gF)(t) = 0$ by the first part of the proof. It now follows that

$$\|TF(t)\| = \|TF(t) - T(gF)(t)\| \leq \|TF - T(gF)\| = \|F - gF\| < \epsilon.$$

Hence we must have $TF(t) = 0$ since ϵ was arbitrary. □

We are set up now to prove the next theorem.

7.5.3. Theorem. *(Cambern) Suppose that $T : C_0(Q, X) \to C_0(K, Y)$ is an isometry, where Y is strictly convex. There exist a subset $\mathcal{B}(T)$ of K, a continuous function φ from $\mathcal{B}(T)$ onto Q, and a continuous map $t :\to V(t)$ from $\mathcal{B}(T)$ into the space of bounded operators on X into Y (with S.O.T.) such that*

$$TF(t) = V(t)F(\varphi(t)) \tag{11}$$

for all $t \in \mathcal{B}(T)$ and $F \in C_0(Q, X)$.

Proof. Given $t \in \mathcal{B}(T)$ (where $\mathcal{B}(T)$ is as defined just above the statement of Lemma 7.5.1), let $\varphi(t) = s$, where $t \in \mathcal{B}(s)$. If $t \in \mathcal{B}(s) \cap B(r)$, then $t \in \mathcal{B}(x, s) \cap \mathcal{B}(z, r)$ for some $s, z \in S(X)$. Let F be an element of $\mathfrak{F}(x, s)$ with $F(r) = 0$. Then $TF(t) = 0$ by Lemma 7.5.2, since $t \in \mathcal{B}(r)$ and $F(r) = 0$. However, $t \in \mathcal{B}(x, s)$ and $F \in \mathfrak{F}(x, s)$ imply that $\|TF(t)\| = \|F\| \neq 0$. This contradiction shows that φ must be well defined. The function φ is clearly onto Q, since each $\mathcal{B}(s)$ is nonempty.

Given $t \in \mathcal{B}(T)$ and $x \in X$, we define $V(t)x = TF(t)$ where $F(\varphi(t)) = x$. If $H(\varphi(t)) = x$, then $(H - F)(\varphi(t)) = 0$, and by Lemma 7.5.2, we have $T(H - F)(t) = 0$, or $TH(t) = TF(t)$. This shows that $V(t)$ is well defined as an operator from X into Y. If $F \in \mathfrak{F}(x, \varphi(t))$, then

$$\|V(t)x\| = \|TF(t)\| \leq \|TF\| = \|F\| = \|x\|,$$

so that $V(t)$ is bounded, and in fact, a contraction.

The definition of $V(t)$ implies that (11) holds. Let us show that φ is continuous on $\mathcal{B}(T)$. If this fails at some $t \in \mathcal{B}(T)$, there is a net $\{t_\beta\}$ converging to t and a neighborhood U of $s = \varphi(t)$ such that for every β_0 there is some $\beta_1 \geq \beta_0$ such that $\varphi(t_{\beta_1}) \notin U$. Now $t \in \mathcal{B}(s)$ means that $t \in \mathcal{B}(x, s)$ for some $x \in S(X)$. There exists $F \in C_0(Q, X)$ such that $F(s) = x$, $\|F\| = \|F(s)\|$, and $F(r) = 0$ for $r \in Q \backslash U$ (e.g., choose $f \in C_0(Q)$ with $\|f\| = 1 = f(s)$ and $f(r) = 0$ for $r \in Q \backslash U$; let $F = f(\cdot)x$). It follows that $\|TF\| = \|TF(t)\|$, and by the continuity of TF, the choice of U, and the choice of $\{t_\beta\}$, we must have

$$1 = \|TF(t)\| = \liminf \|TF(t_\beta)\| = 0.$$

This contradiction establishes the continuity of φ.

The continuity of $t \to V(t)$ in the S.O.T. follows exactly as in the proof of Theorem 7.2.7, since we can find a function F that is constant on a neighborhood of a given s. □

The characterization of the isometry in the previous theorem leads us to make a definition.

7.5.4. DEFINITION. *A linear operator from $C_0(Q,X)$ into $C_0(K,Y)$ is said to be a* generalized weighted composition operator *if there exist a subset K_1 of K, a continuous function φ from K_1 onto Q, and a continuous operator-valued map V from K_1 into $\mathcal{L}(X,Y)$ with the strong operator topology such that $TF(t) = V(t)F(\varphi(t))$ for all $t \in K_1$. In addition, we will say that (X,Y) has the* generalized Banach-Stone property *if every isometry from $C_0(Q,X)$ into $C_0(K,Y)$ is a generalized weighted composition operator.*

Theorem 7.5.3 says that if Y is strictly convex, then (X,Y) has the generalized Banach-Stone property for every Banach space X. It turns out that the only real spaces that are universal for the generalized Banach-Stone property are the strictly convex spaces. We summarize this information in the next theorem.

7.5.5. THEOREM. *(Jeang and Wong) Let Y be a real Banach space. The following two conditions are equivalent.*

(i) *Y is strictly convex.*
(ii) *For every Banach space X, (X,Y) has the generalized Banach-Stone property.*

If Y is a complex space, we still have the implication (i) $\Longrightarrow$ (ii).

PROOF. The implication (i) $\Longrightarrow$ (ii) for Y either real or complex is the conclusion of Theorem 7.5.3.

Hence, suppose Y is real and not strictly convex. There must exist distinct elements e_1 and e_2 of the unit sphere of Y and some λ_0 with $0 < \lambda_0 < 1$ such that $\lambda_0 e_1 + (1-\lambda_0)e_2 \in S(Y)$. It follows that

$$\|\alpha e_1 + \beta e_2\| = \alpha + \beta, \quad \text{for all } \alpha, \beta \geq 0. \tag{12}$$

The above equation holds because it is true that $\|\lambda e_1 + (1-\lambda)e_2\| = 1$ for all $\lambda \in [0,1]$. If this last statement is not true, then there exists $\lambda \in (0,1)$ such that $e_3 = \lambda e_1 + (1-\lambda)e_2$ has norm strictly less than 1. If $\lambda < \lambda_0$, let $\beta = \dfrac{1-\lambda_0}{1-\lambda}$, so that $\lambda_0 e_1 + (1-\lambda_0)e_2 = (1-\beta)e_1 + \beta e_3$. Therefore,

$$1 = \|(1-\beta)e_1 + \beta e_3\| \leq (1-\beta)\|e_1\| + \beta\|e_3\| < (1-\beta) + \beta = 1,$$

a contradiction. Similarly, for $\lambda > \lambda_0$, set $\beta = \dfrac{\lambda_0}{\lambda}$ and obtain

$$1 = \|\lambda_0 e_1 + (1-\lambda_0)e_2\| = \|(1-\beta)e_2 + \beta e_3\| < (1-\beta) + \beta = 1,$$

which is a contradiction once again.

Now we define an operator T from $C(Q, \mathbb{R})$ into $C_0(K, Y)$, where $Q = K = \{1, 2\}$ in the discrete topology, by

$$Tf(1) = \frac{f(1)+f(2)}{2}e_1 + \frac{f(1)-f(2)}{2}e_2, \tag{13}$$

$$Tf(2) = \frac{f(1)+f(2)}{2}(-e_1) + \frac{f(1)-f(2)}{2}e_2. \tag{14}$$

It is clear that

$$\|Tf\| = \max\{\|Tf(1)\|, \|Tf(2)\|\} \leq \left|\frac{f(1)+f(2)}{2}\right| + \left|\frac{f(1)-f(2)}{2}\right|.$$

To show that the inequality above is actually an equality, and that the right-hand side actually equals $\|f\|$, we consider cases.

Case 1. $\|f\| = f(1) \geq |f(2)|$. Then $f(1)+f(2) \geq 0$ and $f(1)-f(2) \geq 0$ so that by (12) we have

$$\begin{aligned}\|Tf(1)\| &= \left\|\frac{f(1)+f(2)}{2}e_1 + \frac{f(1)-f(2)}{2}e_2\right\| \\ &= \frac{f(1)+f(2)}{2} + \frac{f(1)-f(2)}{2} \\ &= f(1) \\ &= \left|\frac{f(1)+f(2)}{2}\right| + \left|\frac{f(1)-f(2)}{2}\right|.\end{aligned}$$

Case 2. $\|f\| = f(2) \geq |f(1)|$. Here we have $f(2)-f(1) \geq 0$ and $f(2)+f(1) \geq 0$ so that again by (12) we see that

$$\begin{aligned}\|Tf(2)\| &= \left\|-\frac{f(1)+f(2)}{2}e_1 + \frac{f(1)-f(2)}{2}e_2\right\| \\ &= \left\|\frac{f(1)+f(2)}{2}e_1 + \frac{f(2)-f(1)}{2}e_2\right\| \\ &= \frac{f(1)+f(2)}{2} + \frac{f(2)-f(1)}{2} \\ &= f(2) \\ &= \left|\frac{f(1)+f(2)}{2}\right| + \left|\frac{f(1)-f(2)}{2}\right|.\end{aligned}$$

The other cases are handled in the same way, and we see that T is an isometry. However, T does not satisfy the conclusion of Lemma 7.5.2 for either of $t = 1$ or $t = 2$ and so cannot be a generalized weighted composition operator. □

We note here that if Y contains a copy of $\ell^\infty(2)$(over the same field as Y), then there are elements y_1, y_2 of $S(Y)$ such that if $y = ay_1 + by_2$, then $\|y\| = \max\{|a|, |b|\}$. One can define, as in Example 7.3.1, $T : C(\{1,2\}, \ell^\infty(2)) \to C(\{1,2\}, Y)$ by

$$TF(1) = \langle F(1), e_1 \rangle y_1 + \langle F(2), e_2 \rangle y_2, \quad \text{and}$$

$$TF(2) = \langle F(2), e_1 \rangle y_1 + \langle F(1), e_2 \rangle y_2,$$

so that T is an isometry. (By $\langle F(j), e_k \rangle$ we mean, of course, the kth coordinate of $F(j)$.) However, T is not a generalized weighted composition operator. Let us record this formally.

7.5.6. THEOREM. *If Y is a Banach space (real or complex) which contains a subspace isometric to $\ell^\infty(2)$ (correspondingly real or complex), then Y is not universal for the generalized Banach-Stone property.*

PROOF. Using the notation introduced prior to the statement of the theorem, let $F \in \mathfrak{F}(e_1, 1)$. Then $\langle F(1), e_1 \rangle = \|F\|$, and $\|TF(1)\| = \|F\|$, so that $1 \in B(e_1, 1)$. Similarly, for $F \in \mathfrak{F}(e_2, 1)$, we get $\|TF(2)\| = \|F\|$. Therefore, $2 \in B(e_2, 1)$ and $B(1) = \{1, 2\}$. The function φ (as defined in the proof of Theorem 7.5.3) is not well defined. It is also easy to see that T cannot satisfy Lemma 7.5.2. $\square$

So far in this section, we have always assumed that the isometries T were defined on all of $C_0(Q, X)$. We want to relax that requirement and consider isometries defined on a closed subspace M of $C_0(Q, X)$. Once again, this parallels the approach taken in Chapter 2. The goal, as always, is to see if we can show that an isometry from such an M onto a subspace N of $C_0(K, Y)$ is some kind of generalized weighted composition operator. We will see that it is necessary to make some assumptions about the subspace M in order to assure the existence of enough functions of the right kind to make the arguments work.

We will need some new notation and concepts in this setting. Given a subspace M of $C_0(Q, X)$, for $s \in Q$ we will let

$$X(s) = \{F(s) \in X : F \in M\}. \tag{15}$$

We will generally treat $X(s)$ as if it were closed. In those cases in which this is not true, we can replace it with its closure, and not cause any significant change in arguments involving it. We will let $\beta(M)$ denote the set of all $s \in Q$ for which $\mathfrak{F}(x, s) \cap M \neq \emptyset$ for all $x \in X(s)$. Furthermore, we will say that $s \in Q$ is a *strong boundary point* for M if for each neighborhood U of s, $x \in X(s)$, and $\epsilon > 0$, there exists $F \in M$ such that $\|F\| = \|F(s)\|$, $F(s) = x$, and $\|F(r)\| < \epsilon$ for all $r \in Q \backslash U$. The set of all strong boundary points for M will be denoted by $\sigma(M)$. Note that Lemma 7.4.3 shows that the strong boundary of a function module is the entire base space Q.

The definition given above of a strong boundary makes sense for a subspace A of $C_0(Q)$ as has been previously discussed in Chapter 2 (see Definition

2.3.9). For a subspace M of $C_0(Q,X)$ and a subspace A of $C_0(Q)$, we write $M \in \mathcal{A}(A)$ to mean that M contains all functions $f(\cdot)x$ where $f \in A$ and $x \in S(X)$. Finally, we will say that M is an A *module* if $fF \in M$ for all $f \in A$ and $F \in M$.

We now suppose that T is an isometry from a subspace M of $C_0(Q,X)$ onto a subspace N of $C_0(K,Y)$. The sets $\mathfrak{F}(x,s)$, $\mathcal{B}(x,s)$, $\mathcal{B}(s)$, and $\mathcal{B}(T)$ are defined as before, except now any function F considered is taken from M.

7.5.7. Lemma. *For each $x \in S(X)$ and $s \in Q$ such that $\mathfrak{F}(x,s) \neq \emptyset$, the set $\mathcal{B}(x,s)$ is not empty.*

Proof. Given $F \in \mathfrak{F}(x,s) \cap M$, the set $K_F = \{t \in K : \|TF(t)\| = \|F\|\}$ is closed and contained in a compact subset of K (since TF vanishes at infinity) and so is itself compact. We will show that the collection $\{K_F : F \in \mathfrak{F}(x,s) \cap M\}$ has the finite intersection property, and so the intersection of the entire collection, the set $\mathcal{B}(x,s)$, will be nonempty.

Hence, suppose $F_1, F_2, \ldots, F_n$ are a given finite collection of elements of $\mathfrak{F}(x,s)$. Define F_0 by $F_0 = \sum_{j=1}^{n} F_j$ and choose $t \in K$ such that $\|TF_0(t)\| = \|TF_0\| = \|F_0\|$. Then

$$\begin{aligned}\sum_{j=1}^{n} \|F_j\| = \sum_{j=1}^{n} \|TF_j\| \geq \sum_{j=1}^{n} \|TF_j(t)\| &\geq \|\sum_{j=1}^{n} TF_j(t)\| \\ &= \|TF_0(t)\| = \|F_0\| \\ &\geq \|\sum_{j=1}^{n} F_j(s)\| \\ &= \|\sum_{j=1}^{n} \|F_j\| x\| = \sum_{j=1}^{n} \|F_j\|.\end{aligned}$$

The inequalities above must actually be equalities, so that we must have $\|TF_j(t)\| = \|F_j\|$ for each j and $t \in \cap_{j=1}^{n} K_{F_j}$. □

We observe here that if $s \in \sigma(M)$, then $\mathfrak{F}(x,s) \cap M \neq \emptyset$ for every $x \in X(s)$, i.e., $\sigma(M) \subset \beta(M)$. Hence, if $M \in \mathcal{A}(A)$ for a subspace A of $C_0(Q)$ and if s is a strong boundary point of A, then $\mathfrak{F}(x,s) \neq \emptyset$ for every $x \in S(X)$, so that $\mathcal{B}(x,s) \neq \emptyset$ for every such x as well. In this case, of course, $\sigma(A) \subset \sigma(M)$. The next lemma is a repeat of Lemma 7.5.2 with $C_0(Q,X)$ replaced by M.

7.5.8. Lemma. *Suppose M is a $C_0(Q)$-module. If $s \in \sigma(M)$, $t \in \mathcal{B}(s)$, and $F(s) = 0$ for some $F \in M$, then $TF(t) = 0$.*

Proof. Since $t \in \mathcal{B}(s)$, there exists $x \in S(X(s))$ such that $t \in \mathcal{B}(x,s)$. Let $F \in M$ with $\|F\| < 1$ and suppose F vanishes on an open neighborhood U of s. Since $s \in \sigma(M)$, we can find $F_1 \in M$ such that $F_1(s) = x$, $1 = \|F_1\|$, and $|F_1(r)| < 1 - \|F\|$ for all $r \in Q \backslash U$. Then $F_1 \in \mathfrak{F}(x,s)$. We define F_2 and

F_3 exactly as in the proof of Lemma 7.5.2, and we follow the same argument to conclude that $TF(t) = 0$.

If $F \in M$ and $F(s) = 0$, we use the fact that M is a $C_0(Q)$-module to produce a continuous g and $gF \in M$ as in the proof of Lemma 7.5.2. Hence, we get $TF(t) = 0$ exactly as before. □

We can now state a theorem that generalizes Theorem 7.5.3. We have enough hypotheses on the subspace M that the proof goes through unchanged.

7.5.9. THEOREM. *Let T be an isometry from a subspace M of $C_0(Q, X)$ onto a subspace N of $C_0(K, Y)$. Assume that Y is strictly convex, and that M is a $C_0(Q)$-module. Then there is a subset $K(T)$ of K, a continuous map φ from $K(T)$ onto the strong boundary $\sigma(M)$ of M, and a continuous map $t :\to V(t)$ from $K(T)$ into the space $\mathcal{L}(X, Y)$ with the S.O.T. such that*

$$TF(t) = V(t)F(\varphi(t)) \ \ \textit{for all} \ \ t \in K(T).$$

If, in addition, we assume that $\sigma(M)$ is a boundary for M, then $K(T)$ is a boundary for N.

PROOF. For each $s \in \sigma(M)$, $B(s) \neq \emptyset$ and we let $K(T) = \bigcup_{s\in\sigma(M)} B(s)$. Now given $t \in K(T)$, we proceed exactly as in the proof of Theorem 7.5.3.

We prove only the last statement. Recall that a subset D of Q is called a *boundary* for M if for each $F \in M$, there is $s \in D$ such that $\|F(s)\| = \|F\|$. Let $G = TF \in N$. By hypothesis, there exists $s \in \sigma(M)$ such that $\|F\| = \|F(s)\|$. Letting $x = F(s)/\|F\|$, we have $x \in S(X)$ and $F \in \mathfrak{F}(x, s)$. For $t \in \mathcal{B}(x, s)$ we have $\|TF(t)\| = \|F\| = \|TF\|$. □

It is straightforward to show that if M is a $C_0(Q)$-module, then the strong boundary of M is equal to $\beta(M)$. Hence we could replace $\sigma(M)$ with $\beta(M)$ in the statements of the two previous results.

The notion of strong boundary is usually given for subspaces A of $C_0(Q)$. Not a great deal is known about the strong boundary even in this case. When $\sigma(A)$ is equal to Q, then A is said to be *extremely regular.* Such subspaces of $C_0(Q)$ exist unless Q is *dispersed.* (See the notes at the end of the chapter. These repeat some similar notes at the end of Chapter 2.) We will examine the case in which $\sigma(M)$ is dense in Q. It is known that $\sigma(A)$ is dense in Q when A is a regular closed subalgebra of $C_0(Q)$. If $M \in \mathcal{A}(A)$ where $\sigma(A)$ is dense in Q, then $\sigma(M)$ will be dense in Q, and this would happen, for example, if A is a regular closed subalgebra. To say that $A \subset C_0(Q)$ is *regular* means that given $s \in Q$ and D a closed subset of Q, there exists $f \in A$ such that $f(s) = 1$ and $f(r) = 0$ for $r \in D$. In fact, a regular closed subalgebra A of $C_0(Q)$ can be shown to be *normal*; that is, given disjoint compact subsets B, D, there is a $g \in A$ such that $g = 0$ on B and $g = 1$ on D. We can use these ideas to get a theorem like the previous one with a slightly stronger conclusion.

7.5.10. THEOREM. *Suppose M, N, T are as in the statement of Theorem 7.5.9 except that we assume $M \in \mathcal{A}(A)$, where A is a normal closed subspace of $C_0(Q)$ with the property that the strong boundary of A is dense in Q. Then* *the conclusion of Theorem 7.5.9 holds for a subset $K_0(T)$ of K which contains $K(T)$, and the function φ maps $K_0(T)$ onto Q. In particular, the conclusion holds if A is a regular closed subalgebra of $C_0(Q)$.*

PROOF. For $s \in Q\backslash\sigma(M)$, we will say that $t \in \mathcal{B}(s)$ if $TF(t) = 0$ whenever $F \in M$ and $F(s) = 0$. We already know that $\mathcal{B}(s) \neq \emptyset$ for $s \in \sigma(M)$. If s is not in the strong boundary, since $\sigma(A)$ is dense, there is a net $\{s_\alpha\}$ from $\sigma(A)$ converging to s. For a given $x \in S(X)$, by Lemma 7.5.7 there is, for each α, some $t_\alpha \in \mathcal{B}(x, s_\alpha)$. This net (or possibly some subnet) must converge to some t in the one-point compactification of K which we denote by $K \cup (\infty)$. If $F \in M$ vanishes in a compact neighborhood U of s, then there is some α_0 such that $s_\alpha \in U$ and therefore $F(s_\alpha) = 0$ for $\alpha \geq \alpha_0$. By Lemma 7.5.8, $TF(t_\alpha) = 0$ for such α and we are left to conclude that $TF(t) = 0$ as well. Furthermore, because A is normal, there exists $g \in C_0(Q)$ such that $g \equiv 1$ on U. For $G = g(\cdot)x$, $\|TG(t_\alpha)\| = 1$ for every $\alpha \geq \alpha_0$. To see this last statement, recall that $G(s_\alpha) = x$, and since $s_\alpha \in \sigma(A)$, there exists some $f \in A$ such that $f(s_\alpha) = 1 = \|f\|$. Let $G_1 = f(\cdot)x$ and observe that $\|TG_1(t_\alpha)\| = 1$ since $t_\alpha \in \mathcal{B}(x, s_\alpha)$. Now if $H = G - G_1$, we have $H(s_\alpha) = 0$ and by Lemma 7.5.8 it follows that $TH(t_\alpha) = 0$. Therefore, $\|TG(t_\alpha)\| = \|TG_1(t_\alpha)\| = 1$ and this holds for all $\alpha \geq \alpha_0$. We conclude that $\|TG(t)\| = 1$ so that $t \neq \infty$. Using arguments like those in the last part of the proof of Lemma 7.5.8 we see that $t \in \mathcal{B}(s)$.

If $t \in \mathcal{B}(s) \cap B(r)$, we need consider only the case where both r and s fail to be in the strong boundary of A. In this case, we let U_1, U_2 be disjoint open neighborhoods of s, r, respectively, whose closures are also disjoint. Choose nets $\{s_\alpha\}$ and $\{r_\beta\}$ from $\sigma(A)$ converging to s and r, respectively, and corresponding nets $\{t_\alpha\}, \{u_\beta\}$ in $\mathcal{B}(x_1, s_\alpha)$ and $\mathcal{B}(x_2, r_\beta)$, where $x_1, x_2 \in S(X)$ and $t_\alpha \to t$, $u_\beta \to t$. Choose $f \in A$ so that $f \equiv 1$ on U_1 and $f \equiv 0$ on U_2. If $F = f(\cdot)x_1$, then there are α_0, β_0 such that

$$\|TF(t_\alpha)\| = \|x_1\| \text{ for } \alpha \geq \alpha_0$$

and

$$\|TF(u_\beta)\| = 0 \text{ for } \beta \geq \beta_0.$$

It follows that

$$\|x_1\| = \|TF(t)\| = 0,$$

and this contradiction shows $\mathcal{B}(s) \cap \mathcal{B}(r) = \emptyset$ for $r \neq s$. We let $K_0(T)$ be the union of the sets $\mathcal{B}(s)$ for $s \in Q$. The remainder of the proof follows as before. □

7.5.11. COROLLARY. *Suppose A, B are both normal closed subspaces of $C_0(Q)$, $C_0(K)$, respectively, such that the the strong boundaries of A, B are dense in Q, K, respectively. Let $M \in \mathcal{A}(A)$ be a $C_0(Q)$-module in $C_0(Q, X)$ and $N \in \mathcal{A}(B)$ a $C_0(K)$-module in $C_0(K, Y)$, where X and Y are both strictly convex. If T is an isometry from M onto N, there exist a homeomorphism*

φ of K onto Q and a continuous map $t :\to V(t)$ from K into $\mathcal{L}(X,Y)$ with S.O.T. such that

$$TF(t) = V(t)F(\varphi(t)) \text{ for all } t \in K, \quad F \in M.$$

Furthermore, $V(t)$ is an isometry if $\varphi(t) \in \sigma(A)$. In particular, the conclusions hold when A, B are regular closed subalgebras.

PROOF. By Theorem 7.5.10, there are continuous functions φ from $K_0(T)$ onto Q and ψ from $K_0(T^{-1})$ onto K, as well as bounded operators $V(t), W(s)$ such that

$$TF(t) = V(t)F(\varphi(t)) \text{ for } t \in K_0(T), \ F \in M;$$
$$T^{-1}G(s) = W(s)G(\psi(s)) \text{ for } s \in K_0(T^{-1}), \ G \in N.$$

If $t \in K$, then $t = \psi(s)$ for some $s \in K_0(T^{-1})$. If $t \notin K_0(T)$, there is some $r \in K_0(T)$ with $r \neq t$ such that $s = \varphi(r)$. Let G be any element of N so that $G(t) = 0$. Then by Lemma 7.5.8 applied to T^{-1}, we necessarily have $T^{-1}G(s) = 0$. Next we get $G(r) = T[T^{-1}G](r) = 0$ by using Lemma 7.5.8 applied again, this time to T. The conclusion is that we cannot separate r, t by elements of N, which is not the case since $N \in \mathcal{A}(B)$ and B is normal. Thus we must conclude that $t \in K_0(T)$ and so $K = K_0(T)$. For the same reasons, we also have $Q = K_0(T^{-1})$.

Now suppose that $s \in Q$ and $F \in M$. It is clear that

$$F(s) = (T^{-1}[TF])(s) = W(s)V(\psi(s))F(\varphi(\psi(s))).$$

If $\varphi(\psi(s)) = r \neq s$, choose F with $F(s) \neq 0 = F(r)$. Then

$$0 \neq F(s) = W(s)V(\psi(s))F(r) = 0,$$

and from this contradiction we conclude that $r = s$. It follows that $\varphi \circ \psi$ is the identity on Q as is $\psi \circ \varphi$ on K. Hence, φ is a homeomorphism as advertised. Furthermore, the operators $W(s), V(t)$ are inverses of each other for $s = \varphi(t)$.

Finally, suppose $t \in \mathcal{B}(s)$ for $s \in \sigma(A)$ and let $x \in S(X)$. Let $F \in M$ such that $F(s) = x$ and $\|F\| = 1$. Since φ is one-to-one, we must have $t \in \mathcal{B}(x, s)$, and because $F \in \mathfrak{F}(x, s)$, it follows that

$$\|V(t)x\| = \|TF(t)\| = \|F\| = 1.$$

This shows that $V(t)$ is an isometry. □

7.6. The Nonsurjective Case for Nice Operators

In this last section we depart a bit from the focus that has been occupying us. We want to return to the subject of *nice* operators, which we first introduced in an earlier section. Here we intend to make use of the extreme point methods that were used in Section 3 of Chapter 2, and also in Section 4 of the current chapter. In doing that, we may as well treat nice operators, since they are defined in terms of action of the conjugate operator on extreme points. Since isometries are also nice, we accomplish some slight generalizations as well. The other main feature of this section is that we operate without assuming that the space Y is strictly convex. This means that it is not so useful to talk in terms of "generalized Banach-Stone properties" for spaces, but rather to concentrate on trying to characterize the operators themselves. Of some interest in itself, is our introduction of the notion of the Choquet boundary of a closed subspace N of a space of vector-valued continuous functions. We begin with this definition.

7.6.1. DEFINITION. *Let K be a locally compact Hausdorff space, and N a closed subspace of $C_0(K, Y)$, where Y is a Banach space. An element $t \in K$ is in the* Choquet boundary *of N, written $ch(N)$, if there exists $y^* \in ext(Y^*)$ such that $y^* \circ \psi_t$ is an extreme point of the unit ball of N^*.*

It is quite easy to show that $ch(N)$ is a boundary of N. For suppose that $G \in N$. By Lemma 7.2.4, there is an extreme point of the unit ball of N^* whose value at G is $\|G\|$. As we have seen before, extreme points of the unit ball of $C_0(K, Y)^*$ are of the form $y^* \circ \psi_t$, where $y^* \in ext(Y^*)$ and ψ_t is the usual evaluation functional. Furthermore, any extreme point for N^* is the restriction to N of an extreme point of $C_0(K, Y)^*$. Hence there exists some $y^* \in ext(Y^*)$ and $t \in K$ such that

$$(y^* \circ \psi_t)(G) = y^*(G(t) = \|G\| \geq \|G(t)\| \geq y^*(G(t)).$$

Thus $t \in ch(N)$ and $\|G(t)\| = \|G\|$.

We want to look at some examples, but first we introduce some useful notation. Given K, Y, N as in Definition 7.6.1, let

$$N^*(t) = \{y^* \in ext(Y^*) : y^* \circ \psi_t \in ext(N^*)\}. \tag{16}$$

For $t \in K$, we have $Y(t)$ as defined by (15), and since an extreme point for the unit ball of $Y(t)^*$ is the restriction to $Y(t)$ of an extreme point of the unit ball of Y^*, we will not always distinguish between them.

In the examples given below, we consider the scalars to be real.

7.6.2. EXAMPLE.

(i) *Let $N \subset C(\{1, 2, 3\}, \ell^2(2))$ be the collection of functions F such that*

$$F(3) = (\langle F(1), e_1\rangle, \frac{1}{2}(\langle F(1), e_2\rangle + \langle F(2), e_2\rangle)).$$

(Here, e_1, e_2 mean the usual unit vectors and $\langle \cdot, \cdot\rangle$ is the inner product in $\ell^2(2)$.) Then $ch(N) = \{1, 2, 3\}$, $N^*(1) = N^*(2) = ext(\ell^2(2)^*)$, and $sp[N^*(3)] = \ell^2(2)^*$, although $N^*(3) \neq ext(\ell^2(2)^*)$.

(ii) *Let $N \subset C(\{1, 2\}, \ell^2(2))$ be the space of functions G such that*

$$G(2) = (\langle G(1), e_1\rangle, \frac{1}{2}\langle G(1), e_2\rangle).$$

Then $ch(N) = \{1, 2\}$ and $N^*(2) = \{\pm e_1^*\}$, which does not span $Y^*(2) = Y^*$.

(iii) *Let φ_1, φ_2 be continuous functions from a locally compact Hausdorff space K to a compact Hausdorff space Q and suppose there is a proper subset Γ of K which is the set of all $t \in K$ such that $\varphi_1(t) = \varphi_2(t)$, and for which $\varphi_1(\Gamma) = Q$. Let T be defined on $C(Q, Y)$ into $C_0(K, Y)$ by*

$$TF(t) = \frac{1}{2}[F(\varphi_1(t)) + F(\varphi_2(t))]$$

for all $F \in C(Q, Y)$. (Here, Y is some given Banach space.) Then T is an isometry, and if N denotes the range of T, we have $ch(N) = \Gamma$, and $N^*(t) = ext(Y^*(t)) = ext(Y^*)$ for all $t \in \Gamma$.

PROOF. In part (i), we have

$$e_2^* \circ \psi_3 = \frac{1}{2}(e_2^* \circ \psi_1 + e_2^* \circ \psi_2)$$

so that $e_2^* \circ \psi_3 \notin ext(N^*)$. However, $e_1^* \in N^*(3)$, as does the element $y^* = (2/\sqrt{5}, 1/\sqrt{5})$. To see this last, recall that given $G \in N$, there must be some $G^* \in ext(N^*)$ so that $G^*(G) = \|G\|$. Choose $G \in N$ such that $G(1) = e_1$, $G(2) = e_2$. Then $G(3) = (1, \frac{1}{2})$ and $\|G\| = \|G(3)\| = \sqrt{5}/2$. The corresponding G^* must be of the form $y^* \circ \psi_3$ where $y^* = (\cos\theta, \sin\theta)$ for some $0 \le \theta \le 2\pi$. Now

$$\cos\theta + \frac{1}{2}\sin\theta = \frac{\sqrt{5}}{2},$$

and it is straightforward to show that we must have $\cos\theta = 2/\sqrt{5}$ and $\sin\theta = 1/\sqrt{5}$. Hence $(2/\sqrt{5}, 1/\sqrt{5}) \in N^*(3)$.

For part (ii), it is easily seen that for any $G \in N$, $\|G\| = \|G(1)\| \ge \|G(2)\|$. Furthermore, $Y(1) = Y(2) = \ell^2(2)$ and $N^*(1) = ext(\ell^2(2)^*)$. Also, $e_1^* \circ \psi_2 = e_1^* \circ \psi_1$ on N, so that $\pm e_1^* \in N^*(2)$. However, if $y^* = (\cos\theta, \sin\theta)$, where $|\cos\theta| < 1$, then $\|y^* \circ \psi_2\| < 1$ when restricted to N. Such y^* cannot be in $N^*(2)$.

Suppose in part (iii) that $t \notin \Gamma$. Then $\varphi_1(t) \ne \varphi_2(t)$ and if $y^* \in ext(Y^*)$, then the definition of the isometry T implies that

$$T^*(y^* \circ \psi_t)(F) = \frac{1}{2}(y^* \circ \psi_{\varphi_1(t)})(F) + \frac{1}{2}(y^* \circ \psi_{\varphi_2(t)})(F)$$

for all $F \in C(Q, Y)$. Hence $T^*(y^* \circ \psi_t)$ is not extreme, and so $y^* \circ \psi_t$ cannot be extreme either. On the other hand, if $t \in \Gamma$, then

$$T^*(y^* \circ \psi_t) = y^* \circ \psi_{\varphi_1(t)},$$

which is extreme, so that $y^* \circ \psi_t$ must also be an extreme point for N^*. □

Our approach in this section will be similar to that of Section 4. We begin, therefore, with a lemma which helps us describe the centralizer of a subspace N of $C_0(K, Y)$.

7.6.3. LEMMA. *Let N be a closed subspace of $C_0(K, Y)$ such that the dimension of the centralizer $Z(Y(t))$ is 1 for every $t \in ch(N)$. Suppose further that either*

(i) *$N^*(t) = ext(Y(t)^*)$ for each $t \in ch(N)$, or*
(ii) *the linear span of $N^*(t)$ is dense in $Y(t)^*$ for each $t \in ch(N)$ and $N \in \mathcal{A}(A)$ where $\beta(A) = Q$.*

Then for each $W \in Z(N)$ there is a scalar-valued function h on K such that

$$WG(t) = h(t)G(t) \text{ for all } t \in ch(N),\ G \in N.$$

PROOF. Suppose that (i) holds and $W \in Z(N)$. Given $t \in ch(N)$, let us define $P(t)$ on $Y(t)$ by

$$P(t)u = WG(t), \text{ where } G \in N \text{ with } G(t) = u.$$

If $H \in N$ with $H(t) = u$, let $y^* \in ext(Y(t)^*)$. By (i), we have $y^* \circ \psi_t \in ext(N^*)$, and since W is a multiplier, there is a scalar $a_W(y^*, t)$ such that

$$W^*(y^* \circ \psi_t) = a_W(y^*, t)(y^* \circ \psi_t). \tag{17}$$

We can apply the same argument as given in the proof of Lemma 7.4.5 to show that $P(t) \in Z(Y(t))$, and since the latter has dimension 1, there is a scalar $h(t)$ so that

$$WG(t) = P(t)u = h(t)u = h(t)G(t).$$

Indeed, this scalar is one of the eigenvalues for W^* and so the values of the function h are bounded by $\|W\|$.

If (ii) holds, we have $N \in \mathcal{A}(A)$, so $Y(t) = Y$ for each t. Again, for $t \in ch(N)$, we define $P(t)u = WG(t)$ for $G \in N$ and $G(t) = u$. Then (17) holds, and as in the previous part, we can show that if $H(t) = u$ also, then $y^*(WG(t)) = y^*(WH(t))$ for all y^* in the span of $N^*(t)$. The density of this set in the dual of $Y(t)$ requires that $WG(t) = WH(t)$ and so $P(t)$ is well defined. Given $u \in Y(t)$, there is, since $\beta(A) = Q$ and $N \in \mathcal{A}(A)$, some $G \in N$ with $\|G\| = \|G(t)\| = \|u\|$. Thus

$$\|P(t)u\| = \|WG(t)\| \leq \|W\|\|G(t)\| = \|W\|\|u\|$$

and $P(t)$ is bounded. To show that $P(t)$ is a multiplier, we will show it is M-bounded. (Recall the remark made just following Definition 7.4.1.)

Since W is a multiplier, it is M-bounded for some bound λ. Recall that this means that there exists $\lambda > 0$ such that for every $G \in N$, WG is contained in every ball which contains $\{\mu G : \mu \in \mathbb{F}, |\mu| \leq \lambda\}$. If $P(t)$ is not λ-bounded, there exist $u, y \in Y$, and $r > 0$ such that $\|\mu u - y\| < r$ for all μ with $|\mu| \leq \lambda$, but $\|P(t)u - y\| \geq r$. Since A is norming, there is an $h \in A$ with $1 = \|h\| = h(t)$ and the functions $G(\cdot) = h(\cdot)u, H(\cdot) = h(\cdot)y$ both in N. Now $\|\mu G - H\| < r$ for all $|\mu| \leq \lambda$, and since W is λ-bounded, we must have $\|WG - H\| < r$. However,

$$\|WG - H\| \geq \|WG(t) - H(t)\| = \|P(t)u - y\| \geq r$$

by the definition of G and H. We conclude that P(t) is M bounded. The remainder of the proof follows as in part (i). □

We are ready now to state and prove a theorem analogous to Theorem 7.5.3. The domain space and the operator are more general than in that theorem, and we do not require strict convexity of the space Y. However, as we have already seen, extra assumptions must be made about the range space.

7.6.4. THEOREM. *Let T be a nice isomorphism from a function module $(Q, (X_s)_{s\in Q}, X)$ onto a subspace N of $C_0(K, Y)$, where $Y(t)$ has trivial centralizer for each $t \in ch(N)$. Suppose that either*

(i) *$N^*(t) = ext(Y(t)^*)$ for each $t \in ch(N)$, or*
(ii) *the linear span of $N^*(t)$ is dense in $Y(t)^*$ for each $t \in ch(N)$ and $N \in \mathcal{A}(A)$ where A is a subspace of $C_0(Q)$ with $\beta(A) = Q$.*

Then there exists a function φ which is continuous from $ch(N)$ onto a dense subset of Q, and for each $t \in ch(N)$ there is a bounded operator $V(t)$ from $X_{\varphi(t)}$ into $Y(t)$ such that for each F in X,

$$TF(t) = V(t)F(\varphi(t)) \text{ for all } t \in ch(N).$$

In case (i) holds, $V(t)$ is nice for each $t \in ch(N)$.

PROOF. The proof will be very similar to that of Theorem 7.4.8. We let h be a continuous function on Q. Then TM_hT^{-1} is in the centralizer of N. By Lemma 7.6.3, there is a bounded function $\tilde{h}$ such that $TM_hT^{-1} = M_{\tilde{h}}$. As in the earlier argument for the proof of Theorem 7.4.8, if we assume that $t \in ch(N)$ and $y^* \in N^*(t)$, then because T is nice, there are $s \in Q$ and $x^* \in ext(X_s^*)$ so that $T^*(y^* \circ \psi_t) = x^* \circ \psi_s$, and we must have $h(s) = \tilde{h}(t)$. It then follows as before that the function $\varphi(t) = s$ is well defined. Now under the assumption in (i), the conclusions we desire follow exactly as in the proof of Theorem 7.4.8.

Let us assume now that (ii) holds. As usual, we define $V(t)$ by $V(t)u = TF(t)$, where $F \in X$, $u \in X_{\varphi(t)}$, and $F(\varphi(t)) = u$. If $F(\varphi(t)) = H(\varphi(t)) = u$ for some $F, H \in X$, then

$$y^*(TF(t)) = y^*(TH(t))$$

for all $y^* \in N^*(t)$. Since the linear span of this set is dense in $Y(t)^*$, the above equation holds for all $y^* \in Y(t)^*$. Hence we have $TH(t) = TF(t)$ and $V(t)$ is well defined. The fact that $TF(t) = V(t)F(\varphi(t))$ comes directly from the definition of $V(t)$. To see that $V(t)$ is bounded, let $u \in X_{\varphi(t)}$ and $F \in X$ be as guaranteed by Lemma 7.4.3. Therefore,

$$\|V(t)u\| = \|TF(t)\| \leq \|TF\| \leq \|F\| = \|u\|.$$

For the continuity of φ we mimic the proof of the same assertion in Theorem 7.5.3. To do that it is necessary to assert the existence of an element $F \in X$ for which $F(s) = x, \|F\| = \|F(s)\| = \|x\| = 1$, and $\|F(r)\| = 0$ for $r \in Q\backslash U$. But such an element exists by Lemma 7.4.3. Lastly, the argument in the proof of Theorem 7.4.8 can be used to show that there is a nonzero element H of X for which

$$TH(t) = V(t)H(\varphi(t)) = 0$$

for all $t \in ch(N)$. Since $ch(N)$ is a boundary, $TH = 0$, and this contradicts the assumption that T is injective. □

We remind the reader that an isometry is also nice, and that $C_0(Q, X)$ is a function module, so that the above theorem holds for an isometry from $C_0(Q, X)$ onto an N with the given properties. Let us now look for an analogue for Theorems 7.5.9 and 7.5.10.

7.6.5. THEOREM. *Let T be a nice isomorphism from a closed subspace M of $C_0(Q, X)$ onto a closed subspace N of $C_0(K, Y)$. Assume that M is an A-module where A separates the points of Q.*

Suppose that $Z(Y(t))$ is trivial and that $N^*(t) = ext(Y(t)^*)$ for each $t \in ch(N)$. Then there is a function φ from $ch(N)$ into $ch(M)$, and for each $t \in ch(N)$ there is a nice operator $V(t)$ from $X(\varphi(t))$ into $Y(t)$ such that

$$TF(t) = V(t)F(\varphi(t))$$

for all $t \in ch(N)$ and $F \in M$. The function φ is continuous at each t for which $\varphi(t) \in \sigma(M)$, and its range is dense in $\sigma(M)$. In particular, if $\sigma(M)$ is dense in Q, then $\varphi(ch(N))$ is dense in Q.

PROOF. Given $t \in ch(N)$ and $y^* \in N^*(t)$ there exists $s \in ch(M)$ and $x^* \in ext(X(s)^*)$ such that

$$T^*(y^* \circ \psi_t) = x^* \circ \psi_s.$$

This holds because T is nice. We define $\varphi(t) = s$. The proof that φ is well defined is the same as for the previous theorem, and the definition of $V(t)$, its boundedness, and its niceness can also be proved exactly as before. If $\varphi(t) \in \sigma(M)$, then one can find the necessary function $F \in M$ to make the argument for continuity at t work as in the previous theorem.

It remains to prove the assertion about the density of $\varphi(ch(N))$. Suppose $s \in \sigma(M)$ is not in the closure of $\varphi(ch(N)$. For each positive integer n there exists $H_n \in M$ such that $\|H_n\| = \|H_n(s)\| = 1$, and $\|H_n(r)\| < 1/n$ for all $r \in \varphi(ch(N))^-$. (This last statement holds because $s \in \sigma(M)$.)

$$\|TH_n(t)\| = \|V(t)H_n(\varphi(t))\| \leq \|H_n(\varphi(t))\| < 1/n$$

for all $t \in ch(N)$. Since each H_n has norm 1, this implies that T^{-1} is unbounded, which is a contradiction. □

In the previous theorem we assumed that N satisfied condition (i) of Lemma 7.6.3. If we assume that N satisfies condition (ii) of that lemma, we get a slightly weaker conclusion.

7.6.6. THEOREM. *Assume that M has the same properties as in Theorem 7.6.5, and that T is a nice isomorphism from M onto $N \subset C_0(K, Y)$. Suppose that $Z(Y(t))$ is trivial and the linear span of $N^*(t) = Y(t)^*$ for each $t \in ch(N)$. If, in addition, $N \in \mathcal{A}(B)$ where B is a subspace of $C_0(K)$ with $\beta(N) = Q$, there exists a function φ from $ch(N)$ into $ch(M)$ and for each $t \in ch(N)$, a linear operator $V(t)$ from $X(\varphi(t))$ into $Y(t)$ such that*

$$TF(t) = V(t)F(\varphi(t))$$

for each $t \in ch(N)$ and $F \in M$. The function φ is continuous and the operator $V(t)$ is a contraction for each t such that $\varphi(t) \in \sigma(M)$. The range of φ is dense in $\sigma(M)$ so that if $\sigma(M)$ is dense in Q, then the range of φ (and so also $ch(M)$) is dense in Q.

PROOF. We get φ well defined as before and for $t \in ch(N)$ and $x \in X(\varphi(t))$ we define $V(t)x = TF(t)$ where $F \in M$ such that $F(\varphi(t)) = x$. If $H \in M$ with $H(\varphi(t)) = x$, we can easily show that

$$y^*(TF(t)) = y^*(TH(t))$$

for all y^* in the linear span of $N^*(t)$. The density of such functionals allows us to show the above equality holds for all $y^* \in Y(t)^*$, so that $TF(t) = TH(t)$. If $\varphi(t) \in \sigma(M)$, and $x \in X(\varphi(t))$, there exists $F \in M$ such that $1 = \|F(\varphi(t))\| = \|F\|$, where $F(\varphi(t)) = x$. Thus

$$\|V(t)x\| = \|V(t)F(\varphi(t))\| = \|TF(t)\| \leq \|F\| = \|x\|.$$

Hence, $V(t)$ is a contraction. The asserted continuity of φ and density of its range follow as in the previous theorem. □

7.6.7. COROLLARY. *If in either of the two previous theorems we assume that M is an A-module where A separates the points of Q (or even $ch(M)$) and $M \in \mathcal{A}(A)$, then the conclusions hold, and in particular, hold for $\sigma(A)$ in place of $\sigma(M)$. If A is a regular closed subalgebra of $C_0(Q)$, then $\varphi(ch(N))$ (and so also $ch(M)$) is dense in Q.*

PROOF. If $M \in \mathcal{A}(A)$, then $\sigma(A) \subset \sigma(M)$. As we have mentioned before, the strong boundary of a regular closed subalgebra is dense. □

We have said nothing in the statements of the last two theorems and the corollary about the continuity of the map $t :\longrightarrow V(t)$. In fact, the continuity of this map, where the operator space is given the S.O.T., can be proved at a given t at which φ is continuous whenever, for a given $x \in X$, there is a function $F \in M$ which is constantly equal to x on a neighborhood of $\varphi(t)$. This would happen, for instance, if $M \in \mathcal{A}(A)$, where A is normal.

The conditions on N in the previous three theorems are perhaps less than satisfying, but they are necessary as we shall see in some examples to follow. An examination of part (iii) of Example 7.6.2 reveals that the subspace N, defined there as the range of a particular isometry, satisfies the hypotheses on N in the statement of Theorem 7.6.5 if Y has trivial centralizer. Hence any isometry from some $C_0(Q,X)$ onto N will be a generalized weighted composition operator. This leads us to state the following theorem.

7.6.8. THEOREM. *Suppose there exists an isometry T from $C_0(Q,X_1)$ onto a subspace N of $C_0(K,Y)$ so that $TF(t)=V(t)F(\varphi(t))$ for all $t\in ch(N)$ where φ is continuous and $V(t)$ is nice for each $t\in ch(N)$. Suppose further that $Y(t)$ has trivial centralizer for each $t\in ch(N)$. If L is any locally compact Hausdorff space, X is any Banach space, and S is a nice isomorphism from $C_0(L,X)$ onto N, then S will be a generalized weighted composition operator whose operator weights are nice.*

PROOF. Given $t\in ch(N)$ and $y^*\in ext(Y(t)^*)$, since $V(t)$ is nice, there exists $x^*\in ext(X_1^*)$ such that $V(t)^*y^*=x^*$. Hence, $x^*\circ\psi_{\varphi(t)}\in ext(C_0(Q,X_1)^*)$, and since T is an isometry, $y^*\circ\psi_t=(T^*)^{-1}(x^*\circ\psi_{\varphi(t)})$ is an extreme point for the unit ball of N^*. Thus $N^*(t)=ext(Y(t)^*)$, and the conclusion follows from Theorem 7.6.5. □

The following shows that the set $\mathcal{B}(T)$ of Theorem 7.5.3 can differ from $ch(N)$.

7.6.9. EXAMPLE. *Let E be two-dimensional real space with norm*

$$\|(a,b)\|_o=\begin{cases}|a|+|b|(\sqrt{2}-1) & \text{if } |b|\le|a|;\\ |b|+|a|(\sqrt{2}-1) & \text{if } |a|\le|b|.\end{cases}$$

The dual norm is given by

$$\|(\alpha,\beta)\|_d=\begin{cases}|\alpha| & \text{if } |\beta|\le(\sqrt{2}-1);\\ \frac{|\alpha|+|\beta|}{2} & \text{if } (\sqrt{2}-1)\le|\beta|\le(\sqrt{2}+1)|\alpha|;\\ |\beta| & \text{if } (\sqrt{2}+1)|\alpha|\le|\beta|.\end{cases}$$

Let $N=sp\{G_1,G_2\}$, where G_1,G_2 are elements of $C_0(\{1,2,3\},E)$ defined by

$$G_1(1)=e_1,\ G_2(1)=e_2$$
$$G_1(2)=-e_1,\ G_2(2)=e_1$$
$$G_1(3)=-e_1,\ G_2(3)=-e_1.$$

Then for $G=aG_1+bG_2\in N$, we have $\|G\|=|a|+|b|$. Also $ch(N)=\{2,3\}$, $E(t)=sp\{e_1\}$ for $t\in ch(N)$, and $ext(E(t)^*)=N^*(t)$ for $t\in ch(N)$. The operator T from $C(\{1\},\ell^1(2))$ to N defined by

$$TF=\langle F(1),e_1\rangle G_1+\langle F(1),e_2\rangle G_2$$

is an isometry. In fact, it has the canonical form with φ defined on all of $\{1,2,3\}$, which is the set $\mathcal{B}(T)$ as defined in Section 5.

Let W be defined on $C_0(\{1,2\})$ to N by

$$Wf=\left[\frac{f(1)+f(2)}{2}\right]G_1+\left[\frac{f(1)-f(2)}{2}\right]G_2.$$

Here, W is an isometry with canonical form where φ is defined only on $ch(N)=\mathcal{B}(W)$.

PROOF. It is straightforward to show that for any $y^* \in ext(E^*)$, $T^*(y^* \circ \psi_1)$ is not an extreme point of $C(\{1\}, \ell^1(2))^*$, so that $1 \notin ch(N)$. However, if $F \in \mathfrak{F}(x,1)$, then $TF(1) = F(1)$ and $\|TF\| = \|TF(1)\|$ so we have $1 \in \mathcal{B}(T)$. □

The space E in the example is not strictly convex, and the isometries have the form guaranteed by Theorem 7.6.5. In fact, the same kind of behavior will occur even if we replace E by $\ell^2(2)$. Note that the set $\mathcal{B}(T)$ is dependent on the operator T, while $ch(N)$ depends only on N.

Let us recall here that if T is a generalized weighted composition operator, then

$$F(\varphi(t)) = 0 \Rightarrow TF(t) = 0. \tag{18}$$

7.6.10. EXAMPLE. *Let E be the space $\mathbb{R}^2$ with norm determined by the unit ball which is the convex set bounded by the unit circle except that the arcs of the circle in the first and third quadrants are replaced by the line segments connecting $(0,1),(1,0)$ and $(-1,0),(0,-1)$ respectively. Then $e_1 = (1,0), e_2 = (0,1)$ are elements of E satisfying (12). Let T be defined as in (13) and (14), and let N be the range of T. It can be shown that $(1,1) \circ \psi_1, (1,1) \circ \psi_2$ are extreme points for the unit ball of N^* so that $ch(N) = \{1,2\}$. However, $N^*(1) = N^*(2) = \{(1,1),(-1,-1)\}$, which does not span E^*. The isometry T is not a generalized weighted composition operator because it does not satisfy (18). In this case, the function φ could be defined, but $V(t)$ cannot be defined correctly since $N^*(1)$ and $N^*(2)$ are not big enough. The space E does have a trivial centralizer since it has no M summands. (Theorem 7.4.14 (iv).)*

7.6.11. EXAMPLE. *Let E be the real space $\ell^1(2)$ and define T as in the previous example. Here we have*

$$T^*((1,1) \circ \psi_1) = \psi_1 \;\; \text{and} \;\; T^*((-1,1) \circ \psi_1) = -\psi_2$$

and the function φ is not well defined. Of course, the centralizer of E is not trivial in this case.

There is one last piece of business we would like to attend to before closing this section. We want to show that we can get a theorem for nice operators that is similar to Theorem 7.5.9. The assumption that the space Y is strictly convex enables us to drop the other conditions on the range space N. It does seem necessary to assume that the space X is reflexive. We begin with a nice operator version of Lemma 7.5.8, which is the main ingredient in the proof of the theorem.

7.6.12. LEMMA. *Suppose that T is a nice isomorphism from a subspace M of $C_0(Q,X)$ onto a closed subspace N of $C_0(K,Y)$ where Y is strictly convex and X is reflexive. Assume further that M is a $C_0(Q)$-module and $\beta(M) = Q$. Suppose that $y^* \circ \psi_t \in ext(N^*)$ and $T^*(y^* \circ \psi_t) = x^* \circ \psi_s$. If $F \in M$ and $F(s) = 0$, then $TF(t) = 0$.*

PROOF. Assume first that $F \in M$ vanishes on a neighborhood U of $s \in Q$, and also that $\|F\| < 1$. Since X is reflexive, $X(s)$ is reflexive as well, and for x^* as given in the statement above, there exists $x \in X(s)$ such that $x^*(x) = \|x\| = 1$. The hypotheses on M guarantee the existence of a function $F_1 \in M$ such that $\|F_1\| = 1$, $F_1(s) = x$, and $\|F_1(r)\| < 1 - \|F\|$ if $r \in Q \backslash U$. Let $G = F + F_1$ and $H = \frac{1}{2}[G + F_1]$. Then $G(s) = H(s) = F_1(s) = x$ and each of the functions has norm 1. Moreover,

$$1 = y^*(TF_1(t)) = x^*(F_1(s)) = x^*(H(s)) = y^*(TH(t))$$
$$= x^*(G(s)) = y^*(TG(t)).$$

Since T is nice, and therefore a contraction, we conclude that

$$\|TG(t)\| = \|TH(t)\| = \|TF_1(t)\| = 1.$$

Note that $TH(t)$ is a proper convex combination of the other two, and since all lie on the surface of the unit ball of the strictly convex space E, they must all be equal to each other. Since

$$TG(t) = TF(t) + TF_1(t),$$

we must conclude that $TF(t) = 0$.

The remainder of the argument is exactly like that of Lemma 7.5.8. □

7.6.13. THEOREM. *Let T be a nice isomorphism as in the statement of Lemma 7.6.12. Then there exists a continuous function φ from $ch(N)$ into Q whose range is dense, and for each $t \in ch(N)$, there is a bounded operator $V(t)$ from $X(\varphi(t))$ to $Y(t)$ such that*

$$TF(t) = V(t)F(\varphi(t)) \text{ for all } t \in ch(N).$$

PROOF. Let $t \in ch(N)$ and suppose

$$T^*(y^* \circ \psi_t) = x^* \circ \psi_s; \quad T^*(z^* \circ \psi_t) = w^* \circ \psi_r.$$

Let $x \in X(r)$ be such that $w^*(x) \neq 0$. There exists $F \in M$ with $F(s) = 0$ and $F(r) = x$. By Lemma 7.6.12 we have $TF(t) = 0$. However,

$$0 \neq w^*(F(r)) = (w^* \circ \psi_r)(F) = z^*(TF(t)) = 0.$$

This contradiction shows that there is a well-defined function φ which pairs t with s.

For $s = \varphi(t)$ and $H(s) = F(s)$, we have $(H - F)(s) = 0$, and by Lemma 7.6.12 again, we get $T(H - F)(t) = 0$. We conclude that the equation

$$V(t)u = TF(t), \text{ where } F(s) = u,$$

describes a well-defined operator from $X(s)$ to $Y(t)$. The rest of the argument follows as in earlier proofs. □

7.7. Notes and Remarks

In his 1950 doctoral thesis, Jerison [**203**] was the first to consider what we might call the Banach-Stone problem for isometries on a continuous vector-valued function space. Given an isometry T from $C(Q, X)$ to $C(K, X)$, where Q and K are compact Hausdorff spaces and X is a Banach space, he wanted to know if Q and K were homeomorphic. The example we have given in the first paragraph is the one given by Jerison to show that the answer is no in general. Example 7.1.1 is just the well-known fact that $\ell^1(2)$ and $\ell^\infty(2)$ are isometric.

The idea of a space Y satisfying the Banach-Stone property was first given by Cambern [**71**] where Q and K were assumed to be compact. We have extended the language to pairs of Banach spaces, and we have allowed the topological spaces to be locally compact. What we, following Behrends [**27**], have called the strong Banach-Stone property was simply called the Banach-Stone property by Cambern. Jeang and Wong [**202**] say that a Banach space Y *solves the Banach-Stone problem* if (X, Y) has the Banach-Stone property (using our language) for every Banach space X.

The book by Behrends [**27**] is the most complete study of the Banach-Stone property which currently exists. The techniques there use the M structure methods and some very deep results are obtained. We have introduced these in a minimal way in Section 4, but the reader looking for the most general results in this theory should consult that reference.

We wish to mention also a couple of other approaches to Banach-Stone type results which we have not treated at all. One is the recognition that Banach-Stone *maps* are related to the notion of *separating* maps as discussed, for example, by Hernandez, Beckenstein, and L. Narici [**177**]. The references in that paper can acquaint the reader with still more work in that area. We could mention again the work of Abramovich [**2**] and others on disjointness preserving operators which we noted in the remarks at the end of Chapter 3.

Strictly Convex Spaces and Jerison's Theorem. The notion of a T-set is due to Myers [**287**], who was Jerison's dissertation supervisor. Jerison [**203**] stated and proved a number of properties of T-sets, and the proof of Lemma 7.2.2 is essentially his. Property (P) was introduced by Cambern [**66**], although we have framed it in terms of pairs of spaces. Lemma 7.2.4 was observed by Cambern, although he did not specify that x^* could be taken to be an extreme point. Lemmas 7.2.5, 7.2.6, and Theorem 7.2.7 are essentially contained in [**66**] and [**71**], although we have organized things a bit differently and considered pairs X, Y of Banach spaces.

Discrepant T-sets were introduced by Jerison [**203**], and he showed that if a space had property (D), then it satisfied the weak Banach-Stone property (our language, not his). In fact, as Cambern [**66**] pointed out, property (D) implies property (P), and therefore the strong Banach-Stone property, but not conversely. Our treatment is extended to the case of pairs, and our arguments are different than those of Jerison (or Cambern). Of course a strictly convex space satisfies property (D), but Jerison gave a different proof that a strictly convex space satisfies the strong Banach-Stone property. The result is a very nice one, and the first big advancement of the Banach-Stone theorem to the vector-valued case.

M-Summands and Cambern's Theorem. Example 7.3.1 was given by Cambern [**66**] to exhibit a space which failed to have the strong Banach-Stone property, and, in fact, to show that any space with an M summand could not have that property. Cambern's language for a space Y with an M summand was that Y was said to *split*. In [**66**], Cambern showed that a finite dimensional space E has the strong Banach-Stone property if and only if it does not split; that is, it has no nontrivial M summand. Later [**71**], he extended that result to any reflexive space. Thus Lemma 7.3.2, Theorem 7.3.3, and Corollary 7.3.4 are due to Cambern in the 1978 paper cited above. Again, we point out that Cambern was not considering pairs X, Y of Banach spaces, and he assumed that the topological spaces Q, K were compact.

Jerison's original example shows that certain continuous function spaces do not satisfy the weak Banach-Stone property. Sundaresan [**361**] shows that the real space $\ell^\infty(n)$ fails to have the weak Banach-Stone property for every $n \geq 2$ (Theorem 7.3.5). It was Cambern, then, who showed that in the three-dimensional case, $\ell^\infty(3)$ is the only space which fails the weak Banach-Stone property [**68**]. This result and more can also be found in [**27**].

Jeang and Wong [**202**] are responsible for Theorem 7.3.7, pointing out that a space containing no copy of real $\ell^\infty(2)$ does *solve the Banach-Stone problem.*

Centralizers, Function Modules, and Behrends' Theorem. The definitions and facts about function modules, multipliers, centralizers, and other aspects of M structure theory are taken directly from Behrends [**27**], as we have previously mentioned. The characterization of extreme points for the dual of a function module (Lemma 7.4.4) is very important. We omitted the proof simply because we had spent so much time in Chapter 2 proving a (less general) version for the $C_0(Q, X)$ spaces. The proof of Lemma 7.4.5 comes from the dissertation of Al-Halees ([**5**] or [**6**]), but the result is also proved in [**27**] as is Corollary 7.4.6.

Nice operators seem to have been introduced by Morris and Phelps [**286**]. A nice operator T from X to Y is an extreme point of the unit ball of $\mathcal{L}(X, Y)$, but the converse is not generally true. The results in this section for nice operators are due to Al-Halees [**5**]. The use of the extreme point techniques lend themselves to the case of nice operators. It was observed by Werner [**383**] that a nice operator T from $C(Q)$ onto $C(K)$ is a weighted composition operator, and Theorem 7.4.10 extends that statement to the vector case. The isometry theorem for function modules was also proved by Behrends [**27**], although he assumed that the centralizers actually consisted of exactly the multiplications by continuous functions. This allows one to show that the function φ is necessarily continuous, because in this case, TM_hT^{-1} induces an isometric algebra isomorphism ω from $C(Q)$ onto a closed self-adjoint subalgebra of $C(K)$. Hence $\omega(h) = h \circ \varphi$ for some continuous function φ by classical results. In the more general case, however, the function φ need not be continuous. As an example of that, consider the subspace X of the bounded scalar-valued functions on $[0,1]$ where $X = \{f : \{t : |f(t)| \geq \epsilon\}$ is finite for every $\epsilon > 0\}$. This space is a Banach function module [**27**, p. 78] and if φ is a one-to-one function from $[0,1]$ onto itself, then $Tf = f \circ \varphi$ is an isometry from X onto X whether φ is continuous or not. Theorem 7.4.10 for the case where T is an isometry is proved in [**27**].

Theorem 7.4.14 is proved (as it seems is nearly everything we have been talking about) by Behrends [**27**] and shows how useful the centralizer condition is in showing spaces have the Banach-Stone property. Thus Corollary 7.4.11 includes most of the known results about the Banach-Stone problem. In addition to the results mentioned in the text, there is the theorem of Lau [**232**] which says that (X, Y) has the strong Banach-Stone property if both Y^* and X^* are strictly convex, and a theorem in [**132**] which says that a space with trivial Hermitian operators must satisfy the strong Banach-Stone property. Both of these results are subsumed by Behrends' theorem since the spaces in question must have trivial centralizers.

The Nonsurjective Vector-Valued Case. It was Holsztynski [**179**] in the scalar case who first considered operators in the Banach-Stone setting which were not necessarily surjective. Cambern [**70**] tackled this problem in the vector-valued case, and Theorem 7.5.3 is the result. Our proof is a bit different than Cambern's, but the important Lemma 7.5.2 is essentially as he proved it. The notion of a generalized weighted composition operator is due to Jeang and Wong [**202**], as is the observation that strict convexity of Y is a necessary condition for (X, Y) to have the generalized Banach-Stone property for all X. (Again, our language, not theirs.)

The results from Lemma 7.5.7 through the remainder of the section are due to Font [**139**], although we have changed some of the proofs, as well as some of the hypotheses a bit. The strong boundary was studied by Araujo and Font [**11**],[**12**], who showed that the strong boundary for a point-separating closed subalgebra A of $C_0(Q)$ is dense in the Šilov boundary of A. It is also known that the Šilov boundary of such a space is all of Q [**142**]. Cengiz [**87**] has studied spaces where the strong boundary is all of Q; he called such a space *extremely regular*. Cengiz showed that such subspaces of $C_0(Q)$ arise, for example, as kernels of nonzero, continuous complex-valued finite regular Borel measures on Q, and that $C_0(Q)$ has proper extremely regular subspaces whenever Q is not *dispersed*. We must also mention that our assumptions about the space M are different than those of Font. For example, the theorem in [**139**, Theorem 1] corresponding to Theorem 7.5.9 assumes only that $M \in \mathcal{A}(A)$ where A is regular. We have assumed that M is a $C_0(Q)$-module and have worked with what we defined as the strong boundary of M. Our assumption seems necessary (at least for us) to prove the second part of Lemma

7.5.8. There we seem to need a function that is one and zero on disjoint compact sets which would seem to suggest that we need a normal space. However, our proof also requires the function to have norm 1. One might define a subspace A of $C_0(Q)$ to be *strongly normal* if given disjoint compact sets D, B, there exists $f \in A$ with $\|f\| = 1$, and $f = 1$ on D, and $f = 0$ on B. However it is known that a strongly normal subspace is necessarily all of $C_0(Q)$ [**241**, p. 178]. As we have pointed out, if one assumes $M \in \mathcal{A}(A)$, one can work with the strong boundary of A instead of $\sigma(M)$, and although the latter is generally larger, there is more known about the former.

Some other papers that are related to material in this section include [**201**], which extends Cambern's theorem to the locally compact case with alternate methods, and [**207**], which makes use of ideas of Jarosz and Pathak [**200**]. Finally, we mention [**371**], which is Vesentini's treatment of the vector-valued case. The reader will recall that we discussed Vesentini's work in the scalar-valued case in some detail in Chapter 2.

The Nonsurjective Case for Nice Operators. This last section consists primarily of work found in [**5**] and [**6**]. The definition of Choquet boundary is a natural generalization of the definition given in Chapter 2, originally due to Novinger [**292**]. Of course the classical definition for scalar-valued functions goes way back, and is discussed in the notes of Chapter 2. The main thrust in this section is to obtain some kind of results for the case where the space Y is not strictly convex. Although it is disappointing to have to put conditions on the range space N, our examples show that it is necessary, if one is to be able to show that the operators have the canonical form. More study of the Choquet boundary in this setting seems to be needed. Theorem 7.6.13 gives a version of Cambern's theorem (Theorem 7.5.3) for nice operators, but we have had to assume the space X is reflexive. It also extends Theorem 7.5.9 to nice operators but with the same restriction. These results can be found in [**6**].

Before closing, we list a few more references that are related to the subject matter of this chapter: [**10**], [**26**], [**28**], [**29**], [**33**], [**72**], [**75**], [**80**], [**159**], [**178**], [**362**], and [**377**].

CHAPTER 8

The Banach-Stone Property for Bochner Spaces

8.1. Introduction

After considering isometries on vector-valued continuous function spaces, it is perhaps natural to turn to an investigation of isometries on vector-valued L^p spaces. Given a measure space (Ω, Σ, μ) and a Banach space X, the space $L^p(\mu, X)$ or just $L^p(X)$ will mean the Banach space of (equivalence classes of) Bochner measurable functions F from Ω to X for which the norm

$$\|F\|_p = \left\{ \int_\Omega \|F(t)\|^p d\mu \right\}^{1/p}$$

is finite. The norm $\|\cdot\|$ denotes the norm on the Banach space X. To say that F is *Bochner measurable* means that there exists a sequence $\{F_n\}$ of simple functions such that $\lim_n \|F_n(t) - F(t)\| = 0$ μ-almost everywhere. (A function F is sometimes called *weakly μ-measurable* if the scalar function $x^* \circ F$ is measurable in the usual sense for each $x^* \in X^*$.) By a simple function is meant a function of the form $F = \sum_{j=1}^n \chi_{A_j} x_j$, where $A_1, \ldots, A_n$ are measurable sets, χ_A is the characteristic function of A, and $x_j \in X$ for each j. Of course, in the case $p = \infty$ we have, as usual,

$$\|F\|_\infty = \text{ess sup}\|F(t)\|.$$

In this chapter we will assume that q is the extended real number conjugate to p, that is,

$$\frac{1}{p} + \frac{1}{q} = 1.$$

The dual space of $L^p(\mu, X)$ contains the space $L^q(\mu, X^*)$ where X^* is the dual of X, and the two are equal if and only if X^* has the Radon-Nikodym property.

Given an isometry U from $L^p(\mu_1, X_1)$ onto $L^p(\mu_2, X_2)$, where X_1, X_2 are Banach spaces and $(\Omega_j, \Sigma_j, \mu_j)$ $(j = 1, 2)$ are σ-finite measure spaces, one can ask the following questions.

(i) Are the two measure algebras isomorphic (in the sense of a regular set isomorphism T as in Chapter 3)?

(ii) Are the two Banach spaces X_j isometrically isomorphic?

(iii) Is there an explicit description of U involving T and isometries between the X_j's?

Using the language given in Chapter 7, we could say that a pair (X_1, X_2) of Banach spaces has the *L^p-Banach-Stone property* (or the Banach-Stone property for Bochner spaces) if the answer to (i) is yes, and the *strong* L^p-Banach-Stone property if the answer is yes to both (ii) and (iii). Of course, in all of this we consider $1 \leq p \leq \infty$, $p \neq 2$. As in Chapter 7, if (X, X) has the property we simply say that X has the property. We note in passing that it is possible to construct simple examples similar to the one in Example 7.1.1 in Chapter 7 in which we have a yes answer for (iii) while it is no for both (i) and (ii). Lamperti's theorem from Chapter 3 shows that both $\mathbb{R}$ and $\mathbb{C}$ have the strong Banach-Stone property for Bochner spaces. Indeed, in this case, we recall from Lamperti's theorem in Chapter 3 that the isometry U (for $1 \leq p < \infty$, $p \neq 2$) is described by

$$Uf(t) = h(t)T_1 f(t), \tag{19}$$

where T_1 is the transformation induced by a regular set isomorphism T from Σ_1 onto Σ_2 ($T_1(\chi_A) = \chi_{TA}$) and h is a function defined on Ω_2 which satisfies

$$\int_{TA} |h|^p d\mu_2 = \int_{TA} \frac{d(\mu_1 \circ T^{-1})}{d\mu_2} d\mu_2 = \mu_1(A) \text{ for } A \in \Sigma_1. \tag{20}$$

In fact, if the isometry is surjective, we can say that the function h above actually satisfies

$$|h(t)|^p = \frac{d(\mu_1 \circ T^{-1})}{d\mu_2} \text{ a.e. on } \Omega_2. \tag{21}$$

This situation is discussed in Section 3.2 and the notes at the end of Chapter 3.

In the vector-valued case, we would expect that the function h in the equations above will be replaced by some kind of measurable operator-valued function.

Following along with the historical development, we begin with a consideration of the case where $X_1 = X_2$ is a separable Hilbert space. We record the first of the results due to Cambern, in which he showed that separable Hilbert spaces have the strong L^p-Banach-Stone property for $1 \leq p < \infty$, $p \neq 2$. He later showed the result true for $p = \infty$ as well, and we will give a proof of that which combines ideas of Cambern and Greim. Also we give the theorem of Greim and Jamison showing that Hilbert spaces have the strong Banach-Stone property for Bochner spaces regardless of separability conditions. Section 3 is devoted to establishing the L^p-Banach-Stone property for Banach spaces other than Hilbert spaces. We will include both Sourour's theorem showing that separable Banach spaces have the strong L^p-Banach-Stone property in the complex case, and Greim's extension, which also covers the case for real spaces. In Section 4 we will examine Lin's work for the case in which $p = 2$. We close with the usual section of notes and remarks.

Note on notation: In what follows we will need to use notation for inner products, semi-inner products, and evaluation of linear functionals at elements of the base space. Previously we have used $\langle\cdot,\cdot\rangle$ for the inner product on a Hilbert space, although the same notation is often used for the evaluation of a linear functional. To avoid confusion, we introduce $\langle\langle x, f\rangle\rangle$ to indicate the evaluation $f(x)$, where f is a linear functional on the space of which x is a member. Hence, if x, y are members of $\mathbb{C}^n$, for example, we would write

$$\langle x, y\rangle = \sum_{j=1}^{n} x(j)\overline{y(j)}$$

and

$$\langle\langle x, y\rangle\rangle = \sum_{j=1}^{n} x(j)y(j).$$

8.2. L^p Functions with Values in Hilbert Space

To begin with, we assume that X is a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and that $(\Omega_j, \Sigma_j, \mu_j)$, $(j = 1, 2)$, are finite measure spaces, and $1 \le p < \infty$, $p \neq 2$.

When X is a Hilbert space there is, of course, a conjugate linear isometry between X and X^*, and we will abuse the notation by just identifying them. Given an isometry U from $L^p(\mu_1, X)$ onto $L^p(\mu_2, X)$, we will make use of the map $(U^*)^{-1}$ defined from $L^q(\mu_1, X)$ to $L^q(\mu_2, X)$ by

$$\int_{\Omega_2} \langle F(t), (U^*)^{-1}(G))(t)\rangle d\mu_2 = \int_{\Omega_1} \langle (U^{-1}(F))(t), G(t)\rangle d\mu_1$$

for $F \in L^p(\mu_2, X)$, $G \in L^q(\mu_1, X)$. Thus $(U^*)^{-1}$ is essentially, but not quite, the Banach space adjoint of U^{-1}.

8.2.1. LEMMA. *If $1 \le p < \infty$ and $p \neq 2$, for $F, G \in L^p(X)$, we have*

$$\|F + G\|_p^p + \|F - G\|_p^p = 2\|F\|_p^p + 2\|G\|_p^p \tag{22}$$

if and only if F and G have a.e. disjoint supports.

We will not prove this lemma, as the proof is a straightforward extension of the same result (Clarkson inequalities and equalities) for scalar-valued functions as given in Chapter 3, or more precisely in [**329**, p. 416].

Assume now that U is a surjective isometry from $L^p(\mu_1, X)$ to $L^p(\mu_2, X)$ for $1 \le p < \infty$, $p \neq 2$. For a scalar-valued function f and $x \in X$, we will let $\tilde{x}$ indicate the function that is identically equal to x and fx will denote the function whose value at any t is $f(t)x$. It follows from (22) and the fact that U is surjective, that the support of $U(\tilde{e})$ is equal to Ω_2 a.e.

8.2.2. LEMMA. *(Cambern) For any $e \in X$ of norm 1, let $F_e = U(\tilde{e})$ and $E_e(t) = F_e(t)/\|F_e(t)\|$ a.e. If $f \in L^p(\mu_1, \mathbb{F}) = L^p(\mu_1)$ (where $\mathbb{F}$ is the scalar field), then $(U(fe))(t) = g(t)E_e(t)$ for some scalar function g, and the mapping $f(t) \to \langle U(fe)(t), E_e(t)\rangle$ is an isometry of $L^p(\mu_1)$ onto $L^p(\mu_2)$.*

PROOF. Let A be a measurable set in Ω_1. Since $U(\tilde{e}) = U(\chi_A e)+U(\chi_{A^c} e)$ (where A^c denotes the complement of A), and U is an isometry, it follows from (22) that $U(\chi_A e)$ and $U(\chi_{A^c} e)$ have disjoint supports, so that $(U(\chi_A e))(t) = \|F_e(t)\| E_e(t)$ on the support of $U(\chi_A e)$. Hence, for any simple function $f \in L^p(\mu_1)$, we have $(U(fe))(t) = g(t)E_e(t)$, where g is a function in $L^p(\mu_2)$ with the same norm as f. Next, let f be an arbitrary member of $L^p(\mu_1)$ and $\{f_k\}$ a sequence of simple functions converging to f in the norm of $L^p(\mu_1)$. It follows that $\|(U(f_k e))(t) - (U(fe))(t)\|^p$ goes to zero in measure so that for some subsequence $\{f_{k_j}\}$, we must have $(U(f_{k_j} e))(t)$ goes to $(U(fe))(t)$ almost everywhere.

From our observations above, we see that for almost all t, the elements $(Uf_{k_j} e))(t)$ of X lie in the one-dimensional subspace of X spanned by $E_e(t)$, and we conclude that $(U(fe))(t)$ must also lie in that space, so that we have $(U(fe))(t) = g(t)E_e(t)$ for some $g \in L^p(\mu_2)$ with $\|f\|_p = \|g\|_p$. This shows that the map is an isometry of $L^p(\mu_1)$ into $L^p(\mu_2)$. To see that the map is onto, suppose we are given a function of the form $g(t)E_e(t)$ where $g \in L^p(\mu_2)$. Let $\{e_n\}$ be an orthonormal basis for X with $e_1 = e$. Let $F(t) = \sum_n f_n(t)e_n$ be the element of $L^p(\mu_1, X)$ which maps onto $g(t)E_e(t)$ under U. Let $F_0(t) = \sum_{n\geq 2} f_n(t)e_n$, which belongs to $L^p(\mu_1, X_0)$, where X_0 is the subspace of X spanned by $\{e_n : n \geq 2\}$. The vector-valued simple functions of the form $G = \sum_{j=1}^{r} \chi_{A_j} u_j$, $u_j \in X_0$ are dense in $L^p(\mu_1, X_0)$. It may be shown (although we omit the proof) that for all such G, $\langle U(G)(t), E_e(t)\rangle = 0$ a.e., and it follows that $\langle U(F_0)(t), E_e(t)\rangle = 0$ a.e. From this we are able to conclude that $U(f_1 e)(t) = g(t)E_e(t)$ a.e. □

Let us also recall that by a regular set isomorphism, we mean a set map T from Σ_1 to Σ_2 defined modulo null sets and satisfying

(i) $T(\Omega_1 \backslash A) = T\Omega_1 \backslash TA)$ for all $A \in \Sigma_1$;
(ii) $T(\cup_1^\infty A_n) = \cup_1^\infty TA_n$ for disjoint $A_n \in \Sigma_1$;
(iii) $\mu_2(TA) = 0$ if and only if $\mu_1(A) = 0$.

As we have previously mentioned, the set transformation induces an operator (which we will henceforth denote by the same notation T) which is defined on characteristic functions χ_A by $T(\chi_A) = \chi_{TA}$ and extended in the standard way.

8.2.3. LEMMA. *(Cambern) Let $\{e_n : n = 1, 2, \cdots\}$ be a fixed orthonormal basis for X, and for each n define F_n, E_n by $F_n = U(\tilde{e_n}), E_n(t) = F_n(t)/\|F_n(t)\|$. Then there exists a regular set isomorphism T from Σ_1 onto Σ_2 and a fixed scalar function h defined on Ω_2 and satisfying (21), such that for all $n = 1, 2, \cdots$ and for all $f \in L^p(\mu_1)$, we have*

$$U(fe_n)(t) = h(t)(T(f))(t)E_n(t),$$

where T is the operator induced by the set transformation of the same name.

PROOF. If e_n, e_m are two elements of the given orthonormal basis, it follows from Lemma 8.2.2 and Lamperti's theorem mentioned in the introduction that there are linear transformations T_n, T_m induced by regular set isomorphisms and scalar functions h_n, h_m defined on Ω_1 so that for any $f \in L^p(\mu_1)$and $t \in \Omega_2$, we have

$$U(fe_j)(t) = h_j(t)T_j(f)(t)E_j(t), \quad j = n, m. \tag{23}$$

The goal is to show that $h_n = h_m$ and $T_n = T_m$ modulo sets of measure zero.

Suppose $A \in \Sigma_1$ and let $F_{m,n} = U[(\tilde{e_m} + \tilde{e_n})/\sqrt{2}]$ and $E_{m,n}(t) = F_{m,n}(t)/\|F_{m,n}(t)\|$. Again from Lemma 8.2.2 and Lamperti's theorem, we obtain a scalar function and regular set isomorphism $T_{m,n}$ such that

$$U[\chi_A(e_m + e_n)/\sqrt{2}](t) = h_{m,n}(t)\chi_{T_{m,n}(A)}(t)E_{m,n}(t) \text{ a.e. for } t \in \Omega_2. \tag{24}$$

We need the fact that for any unit vector e,

$$((U^*)^{-1}(\tilde{e}))(t) = \|U(\tilde{e})(t)\|^{p-1}U(\tilde{e})(t)/\|U(\tilde{e})(t)\| \text{ a.e.} \tag{25}$$

Using this fact (whose proof we omit) and the linearity of U we can establish

$$\begin{aligned}(U^*)^{-1}[(\tilde{e_m} + \tilde{e_n})/\sqrt{2}](t) &= \|(F_m(t) + F_n(t))/\sqrt{2}\|^{p-1}E_{m,n}(t) \\ &= \|(F_m(t) + F_n(t))/\sqrt{2}\|^{p-1}(\|F_m(t)\|E_m(t) \\ &\quad + \|F_n(t)\|E_n(t))/\|F_m(t) + F_n(t)\|.\end{aligned} \tag{26}$$

Using (25) again with the linearity of $(U^*)^{-1}$, we also have

$$\begin{aligned}(U^*)^{-1}[(\tilde{e_m} + \tilde{e_n})/\sqrt{2}](t) &= \|F_m(t)\|^{p-1}E_m(t)/\sqrt{2} \\ &\quad + \|F_n(t)\|^{p-1}E_n(t)/\sqrt{2}.\end{aligned} \tag{27}$$

Now $E_m(t)$ and $E_n(t)$ are orthogonal and so linearly independent. Using this together with (26) and (27) and equating appropriate coefficients, we infer that

$$E_{m,n}(t) = \frac{E_m(t)}{\sqrt{2}} + \frac{E_n(t)}{\sqrt{2}}. \tag{28}$$

If we put this together with the equations (23) applied to χ_A as well as equation (24), make use again of the independence of $E_m(t)$ and $E_n(t)$ and equate coefficients, we obtain

$$h_m(t)\chi_{T_m(A)}(t) = h_{m,n}(t)\chi_{T_{m,n}(A)}(t) = h_n(t)\chi_{T_n(A)}(t) \tag{29}$$

for almost all t. Since this holds for every measurable A, we must conclude that $h_n = h_m$ and $T_n = T_m$ modulo sets of measure zero. If we let $T = T_1$ and $h = h_1$, we get the desired statement as given in the lemma. □

The groundwork is now laid for the statement of the main theorem. A function V defined on the measure space to the bounded operators on X is said to be *stongly measurable* if $V(t)e$ is a Bochner measurable function of t for all

$e \in X$. Furthermore, if T is the linear isometry induced by a measurable set transformation, we extend it to vector-valued functions $F = \sum_n f_n e_n$ by

$$T(F)(t) = \sum_n T(f_n)(t)e_n. \tag{30}$$

8.2.4. THEOREM. *(Cambern) Let U be an isometry of $L^p(\mu_1, X)$ onto $L^p(\mu_2, X)$ and let $\{e_n : n = 1, 2, \cdots\}$ be an orthonormal basis for the separable Hilbert space X. Then there exists a regular set isomorphism T from the σ-algebra Σ_1 onto Σ_2 (defined modulo null sets), a scalar function defined on Ω_2 satisfying (21), and a strongly measurable operator-valued function V defined on Ω_2, where $V(t)$ is a surjective isometry of X for almost all t, such that for $F \in L^p(\mu_1, X)$,*

$$U(F)(t) = V(t)h(t)T(F)(t),$$

where $T(F)$ is as defined in (30). Conversely, every transformation U of the above form is a surjective isometry.

PROOF. Suppose that U is an isometry as described in the statement of the theorem. For $t \in \Omega_2$ we define $V(t)$ on the basis vectors of X by $V(t)(e_n) = E_n(t)$, where the E_n are as given in Lemma 8.2.3. We extend $V(t)$ linearly to all of X. As we have previously observed, the $E_n(t)$'s form an orthonormal set in X, and so $V(t)$ is an isometry for almost all t. Now we apply Lemma 8.2.3 to obtain the induced transformation T. If X is finite-dimensional, the assertions of the theorem would follow from the lemma. Hence, let us assume that X is infinite-dimensional.

Let $F(t) = \sum_n f_n(t)e_n$ be an element of $L^p(\mu_1, X)$ and let $\{F_n\}$ denote the sequence of partial sums which converges a.e. to F. We can apply the dominated convergence theorem to $\{\|F_N(t) - F(t)\|^p\}$ to see that $\|F_n - F\|_p \to 0$. Hence, $U(F) = \lim_N UF_N$ in $L^p(\mu_2, X)$ so that a subsequence of $\{U(F_N)\}$ converges to $U(F)$ a.e. Now $\|F(t)\|^2 = \sum_n |f_n(t)|^2 = \lim_n \sum_{n=1}^N |f_n(t)|^2$ and we have

$$\begin{aligned}
|T(\|F\|)|^2 = T(\|F\|^2)(t) &= \lim_N \left(T\left(\sum_{n=1}^N |f_n|^2 \right) \right)(t) \\
&= \lim_N \sum_{n=1}^N |T(f_n)(t)|^2 = \sum_n |T(f_n)(t)|^2 \\
&= \|T(F)(t)\|^2.
\end{aligned} \tag{31}$$

Since $V(t)$ is a.e. norm preserving, we obtain from (31) that

$$V(t)h(t)T(F)(t) = \lim_N V(t)h(t)T(F_N)(t) = \lim_N U(F_N)(t)$$

exists in X for almost all $t \in \Omega_2$. It follows that

$$U(F)(t) = V(t)h(t)T(F)(t) \tag{32}$$

as claimed in the assertion of the theorem. Since U is onto, it follows that $V(t)$ must map X onto itself for almost all t. For, if D is a countable dense subset of X, then for any $y \in D$, $\tilde{y} = UF$ for some F, and it follows from (32) that y is in the range of $V(t)$ for almost all t. Since the countable set D is thus in the range of $V(t)$ for almost all t, we see that $V(t)$ is surjective almost everywhere.

The converse is straightforward to see, using the calculation in (31).

□

The case where $p = \infty$ is not covered in the previous theorem and we want to take care of that next. Cambern did this for the case of separable Hilbert spaces, but we are going to take a different approach which will culminate in a removal of the separability assumption.

The first result we want to exhibit is due to Greim and is of considerable interest in its own right. It also has implications for the case in which the space X is not necessarily a Hilbert space. We will make use of the notion of centralizer of a Banach space which was central to the discussions in Chapter 7. Recall that $Z(X)$ denotes the centralizer of X, the set of all multipliers or M-bounded operators on X which have an "adjoint." In the proof below we will make use of the notion of a lifting. A linear map σ from $L^\infty(\mu, X)$ into the space $M^\infty(\mu, X)$ of bounded (Bochner) measurable functions with the supremum norm is called a *lifting* if for each (equivalence class) F in $L^\infty(\mu, X)$, σF is in the equivalence class $[F]$. If X is a Banach dual, a lifting σ may be chosen so that $\sigma([\tilde{x}]) = \tilde{x}$. (See the notes at the end of the chapter for references.)

8.2.5. THEOREM. *(Greim) If X is a dual Banach space, then the space $L^\infty(\mu, Z(X))$ is isometrically isomorphic to $Z(L^\infty(\mu, X))$.*

PROOF. We will assume that our Banach spaces are over the real field. For if the field is complex, and $X_{\mathbb{R}}$ denotes the underlying real space of X, then $L^\infty(\mu, Z(X)) = L^\infty(\mu, Z(X_{\mathbb{R}})) + iL^\infty(\mu, Z(X_{\mathbb{R}}))$ and $Z(L^\infty(\mu, X)) = Z(L^\infty(\mu, X_{\mathbb{R}})) + iZ(L^\infty(\mu, X_{\mathbb{R}}))$.

Suppose H is an element of $L^\infty(\mu, \mathcal{L}(X))$, where, as usual, $\mathcal{L}(X)$ denotes the bounded operators on X. For $F \in L^\infty(\mu, X)$, let $M_H F$ be the element of $L^\infty(\mu, X)$ defined by

$$(M_H F)(t) = H(t)(F(t)).$$

It is easy to see that M_H is a bounded operator from $L^\infty(\mu, X)$ to itself with $\|M_H\| \le \|H\|$. In fact, one can show that the mapping $H \to M_H$ is isometric on the dense subspace of countably valued functions. Now the question is whether this map takes a $Z(X)$-valued operator H into $Z(L^\infty(\mu, X))$.

Assume, then, that $H(t) \in Z(X)$ for each t. Since the scalars are real, this means that $H(t)$ is M bounded for each t. We must show that M_H is M bounded; that is, there exists a constant $\lambda > 0$ such that for $F, G \in L^\infty(\mu, X)$, $\|G \pm \lambda F\| \le \alpha$ implies that

$$\|G \pm M_H F\| \le \alpha, \quad \text{for} \quad \alpha > 0.$$

Choose $\lambda = \|M_H\| = \|H\|$ and assume that $\|G - \lambda F\| \leq \alpha$. Then $\|G(t) - \lambda F(t)\| \leq \alpha$ a.e., and since each $H(t)$ is M bounded with $\|H(t)\| \leq \lambda$ a.e., we conclude that $\|G(t) - H(t)F(t)\| \leq \alpha$ a.e. and M_H is in $Z(L^\infty(\mu, X))$.

On the other hand, suppose that R is a norm 1 element of $Z(L^\infty(\mu, X))$ and σ is a lifting as mentioned just prior to the statement of the theorem. We define, for each $t \in \Omega$, an operator $H(t)$ on X by

$$H(t)x = \sigma(R[\tilde{x}])(t).$$

Then $H(t)$ is a linear contraction and the mapping $t \to H(t)$ is strongly measurable in the sense that for all $F \in L^\infty(\mu, X)$, $t \to H(t)F(t)$ is Bochner measurable. We want to show that $H(t)$ is M bounded. Let $\alpha > 0$ be such that $\|y \pm x\| \leq \alpha$. It follows that $\|[\tilde{y}] \pm [\tilde{x}]\| = \|y \pm x\| \leq \alpha$, so that $\|[\tilde{y}] \pm R[\tilde{x}]\| \leq \alpha$ since R is M bounded. From this we get

$$\begin{aligned}\|y \pm H(t)x\| &= \|y \pm \sigma(R[\tilde{x}])(t)\| \\ &= \|\tilde{y}(t) \pm \sigma(R[\tilde{x}])(t)\| \\ &\leq \|\sigma[\tilde{y}] \pm \sigma R[\tilde{x}]\| \\ &= \|[\tilde{y}] \pm [R[\tilde{x}]]\|.\end{aligned}$$

Now since X is a dual space, the strong and norm topologies on $Z(X)$ coincide, and it may be shown under those circumstances that the mapping $t \to H(t)$ is measurable and therefore describes an element H of $L^\infty(\mu, Z(X))$. The operator M_H is equal to R on the constant functions and the two operators commute with the characteristic projections $t \to \chi_A(t)$. Hence they coincide on the dense set of countably valued functions, which is enough to conclude that $M_H = R$ on all of $L^\infty(\mu, X)$. □

We are ready now to prove the version of Theorem 8.2.4 for $p = \infty$. The following lemma will be helpful.

8.2.6. LEMMA. *(Cambern) Let $\{e_n\}$ denote an orthonormal basis for the separable Hilbert space X_1 and suppose that U is an isometry from $L^\infty(\mu_1, X_1)$ onto $L^\infty(\mu_2, X_2)$. Then $\{U\tilde{e_n}(t)\}$ is an orthonormal set in X_2 for almost all t.*

PROOF. First we note that for any $n \in \mathbb{N}$, the function $\tilde{e_n}$ is an extreme point in $L^\infty(\mu_1, X_1)$, so that $U\tilde{e_n}$ is an extreme point in $L^\infty(\mu_2, X_2)$ and we have $\|U\tilde{e_n}(t)\| = 1$ for almost all $t \in \Omega_2$. If k, m are distinct positive integers, it follows from the polarization identity for inner products (and assuming the spaces are complex) that for almost all t,

$$\langle U\tilde{e_k}(t), U\tilde{e_m}(t)\rangle = \frac{1}{4}\sum_{j=1}^{4} i^j \|U\tilde{e_k}(t) + i^j U\tilde{e_m}(t)\|^2. \tag{33}$$

However, the elements $\tilde{e_k} + i^j\tilde{e_m}$ are all extreme points of the ball of radius $\sqrt{2}$ centered at the origin of $L^\infty(\mu_1, X_1)$ from which we conclude that $\|(U[\tilde{e_1} + i^j\tilde{e_m})(t)\| = \sqrt{2}$ almost everywhere, $j = 1, 2, 3, 4$. Hence equation (33) is 0 almost everywhere. (Note, in the real case we have an alternate form for (33).) □

We mention here that if we have a regular set isomorphism T from Σ_1 to Σ_2, it induces an isometry (still denoted by T) from $L^\infty(\mu_1, X_1)$ onto $L^\infty(\mu_2, X_1)$ by means of the equation

$$T(\chi_A x) = \chi_{TA} x.$$

The map T can be extended since the countably valued functions are dense in Bochner L^∞ spaces.

8.2.7. THEOREM. *(Cambern, Greim) Let $X_j, j = 1, 2$ be separable Hilbert spaces and U a surjective isometry from $L^\infty(\mu_1, X_1)$ to $L^\infty(\mu_2, X_2)$. There is a regular set isomorphism T from Σ_1 to Σ_2 (defined modulo null sets) and a strongly measurable operator-valued function $t \to V(t)$ such that for $F \in L^\infty(\mu_1, X_1)$,*

$$UF(t) = V(t)T(F)(t), \tag{34}$$

where $V(t)$ is almost everywhere a surjective isometry of X_1 onto X_2 and T is the operator induced by the set isomorphism.

PROOF. The isometry U determines an isometry between the centralizers of $L^\infty(\mu_1, X_1)$ and $L^\infty(\mu_2, X_2)$ by means of the transformation $\Phi(R) = URU^{-1}$. By Theorem 8.2.5 and the fact that the centralizer of a Hilbert space is one-dimensional, this isometry between centralizers may be interpreted as an isometry of $L^\infty(\mu_1)$ onto $L^\infty(\mu_2)$. These scalar-valued spaces can be regarded as $C(K)$ spaces for a compact, Hausdorff space K, and by the classical Banach-Stone theorem, the restriction of Φ to the idempotent elements in $L^\infty(\mu_1)$ sets up a regular set isomorphism T from Σ_1 to Σ_2 which satisfies $U(\chi_A x) = \chi_{TA} U(\tilde{x})$ for each $A \in \Sigma_1$ and $x \in X_1$. Now, $S = UT^{-1}$ defines an isometry of $L^\infty(\mu_2, X_1)$ onto $L^\infty(\mu_2, X_2)$ and we desire to show that for any $G \in L^\infty(\mu_2, X_1)$,

$$SG(t) = V(t)G(t)$$

almost everywhere, where V is a strongly measurable function on Ω_2 with values as isometries of X_1 onto X_2.

Assuming that $\{e_n\}$ is an orthonormal basis for X_1, and for $t \in \Omega_2$, let us define $S(t)$ by

$$S(t)(e_n) = (\sigma(S\tilde{e_n})(t).$$

We extend this definition linearly to the span of the e_n's, and by Lemma 8.2.6 we conclude that $S(t)$ is an isometry for almost all t since it maps an orthonormal basis to an orthonormal basis. Let us denote by $V(t)$ the extension of $S(t)$ to all of X_1 for all t outside some null set Ω_0. It now follows that $t \to V(t)$ is strongly measurable and

$$\hat{S}G(t) = V(t)G(t) \text{ for all } t \notin \Omega_0$$

defines an isometry on $L^\infty(\mu_2, X_1)$ into $L^\infty(\mu_2, X_2)$. This isometry agrees with the isometry S on all constant functions $\tilde{x}$ and therefore on the dense set of countably valued functions. Hence, $S=\hat{S}$. Now let y be an element of a countable dense subset D of X_2 and suppose that $F = S^{-1}(\tilde{y})$. We must have

$$V(t)F(t) = \hat{S}F(t) = SF(t) = y \text{ a.e.}$$

and it follows that the dense set D is in the range of $V(t)$. Thus $V(t)$ is a surjective isometry almost everywhere. We can define $V(t)$ to be any surjective isometry from X_1 to X_2 on the excluded set of measure zero. □

We have thus shown that separable Hilbert spaces have the strong L^p-Banach-Stone property for all $p \neq 2, 1 \leq p \leq \infty$. Indeed, it is the case that pairs (X_1, X_2) of separable Hilbert spaces have the strong Banach-Stone property, although this is of little consequence since such pairs are isometric anyway. However, we are going to see that such pairs have the property if it is only assumed that one is separable. Our proofs have used the fact that the measure spaces are finite, although the results also hold for σ-finite measures.

Now it is possible to remove the separability condition, and we will give a sketchy account of that. First we state without proof a theorem due to Greim that shows how to get an appropriate regular set isomorphism. It is proved using the notions of L^p projections and integral modules.

8.2.8. THEOREM. *(Greim) If $U : L^p(\mu_1, X_1) \to L^p(\mu_2, X_2)$ is a surjective isometry, where X_1, X_2 are nonzero Hilbert spaces, then there is a regular set isomorphism T of Σ_1 onto Σ_2 (modulo null sets) such that*

$$\chi_{TA} \circ U = U \circ \chi_A$$

for all $A \in \Sigma_1$.

It follows from this that pairs of Hilbert spaces (separable or not) have the Banach-Stone property for Bochner spaces. In fact, Greim and Jamison have shown that such pairs have the strong Banach-Stone property and we want to indicate how that goes.

8.2.9. PROPOSITION. *(Greim and Jamison) Let $1 \leq p \leq \infty$ and suppose that U is a surjective isometry from $L^p(\mu_1, X_1)$ to $L^p(\mu_2, X_2)$, where X_1 and X_2 are infinite-dimensional Hilbert spaces. Suppose further that there is a regular set isomorphism T from Σ_1 onto Σ_2 such that*

$$U \circ \chi_A = \chi_{TA} \circ U.$$

If W_0 is an infinite-dimensional closed subspace of X_1, then there are closed subspaces W_j of X_j, $j = 1, 2$, such that

(i) *$W_0 \subset W_1$,*
(ii) *$U(L^p(\mu_1, W_1) = L^p(\mu_2, W_2)$,*
(iii) *dens W_1 = dens W_2 = dens W_0,*

where dens denotes the density character of the subspace, that is, the smallest cardinal of a dense subset.

PROOF. Although the proposition is true for σ-finite measure spaces, we will assume the measures to be finite. Let $W_1^1 = W_0$ and let W_2^1 be an arbitrary separable subspace of X_2. Also let A_1^1, A_2^1 be dense subsets of W_1^1, W_1^2, respectively, with cardinalities equal to the appropriate density characters. If $A_j^n, W_j^n, j = 1, 2$ have been chosen, for every $x \in A_1^n$, choose a separable subspace W_x of X_2 such that almost all values of $U\tilde{x}$ are in W_x. (This is possible because Bochner measurable functions are essentially separably valued.) Define $W_2^{n+1} = \overline{sp}(W_2^n \cup \bigcup\{W_x : x \in A_1^n\})$. Then dens $W_2^{n+1} \leq$ max {dens W_2^n,card(A_1^n)}. Choose an appropriate $A_2^{n+1} \subset W_2^{n+1}$. Repeat this construction with U replaced by U^{-1} in order to find W_1^{n+1} and A_1^{n+1}. Then dens $W_0 \leq$ dens $W_1^{n+1} \leq$ max {dens W_1^n, card A_2^{n+1}} = dens W_0. For each $j = 1, 2$, we define W_j by

$$W_j = \overline{sp}(\cup\{W_j^n : n \in \mathbb{N}\}).$$

It is clear that dens $W_1 =$ dens W_0, and dens $W_2 \leq$ dens W_0 and we wish now to show that $U(L^p(\mu_1, W_1)) = L^p(\mu_2, W_2)$. To this end, we will find two sets A_1, A_2 with $\overline{sp}(A_j) = W_j$ $(j = 1, 2)$ such that

(i) $U(\tilde{x}) \in L^p(\mu_2, W_2)$ for all $x \in A_1$;
(ii) $U^{-1}(\tilde{y}) \in L^p(\mu_1, W_1)$ for all $y \in A_2$.

The sets $A_1 = \cup A_1^n$ and $A_2 = \cup A_2^n$ have the required properties. For if $x \in A_1$, then $x \in A_1^n$ for some n, and $U(\tilde{x})$ has almost all values in $W_2^{n+1} \subset W_2$. A similar argument verifies (ii). Hence (i) and (ii) are true for all countably valued functions taking values in A_1 and A_2, respectively, and since these functions are dense in $L^p(\mu_j, W_j)$ for $j = 1, 2$, respectively, we have the desired result.

It remains to show that dens $W_2 \geq$ dens W_1. Let $\hat{U}$ denote the restriction of U to $L^p(\mu_1, W_1)$ which maps onto $L^p(\mu_2, W_2)$. If the dimension of W_2 is infinite, we can apply what we have just shown above to $\hat{U}^{-1}$ and W_2 in place of U and W_0 to find a subspace W_3 of W_1 with $\hat{U}^{-1}(L^p(\mu_2, W_2) = L^p(\mu_1, W_3)$ and dens $W_3 \leq$ dens W_2. However, since we also have $\hat{U}^{-1}(L^p(\mu_2, W_2) = L^p(\mu_1, W_1)$, we conclude that $W_3 = W_1$, so that dens $W_1 \leq$ dens W_2.

To complete the proof, we must show that W_2 is infinite-dimensional. Hence, suppose it is finite-dimensional. Let us repeat the original construction with U replaced by $\hat{U}$, W_0 any separable subspace of W_1, and taking W_2 for W_2^1. This gives us a separable $W_3 \subset W_1$ such that

$$\hat{U}(L^p(\mu_1, W_3)) = L^p(\mu_2, W_2).$$

We must have $W_3 = W_1$, so that W_1 is separable. Since we have a description of isometries on L^p with separable Hilbert space values (Theorem 8.2.4, Theorem 8.2.7) we know that the infinite-dimensional space W_1 is isometric with W_2, which contradicts the fact that W_2 is finite- dimensional.

□

We remark here that if X_1 were assumed to be separable, then we can take $W_1 = X_1$ in the above proof, so that from item (ii) in the statement of the proposition, and the fact that U is surjective, $X_2 = W_2$ is also separable. Hence we can state the following result.

8.2.10. COROLLARY. *If either X_1 or X_2 is a separable Hilbert space, then (X_1, X_2) has the strong L^p-Banach-Stone property for $1 \leq p \leq \infty, p \neq 2$.*

The next theorem says that pairs of Hilbert spaces (separable or not) have the strong L^p-Banach-Stone property.

8.2.11. THEOREM. *Let $(\Omega_j, \Sigma_j, \mu_j)$ be σ-finite measure spaces for $j = 1, 2$, suppose X_1, X_2 are Hilbert spaces, and for $1 \leq p \leq \infty, p \neq 2$, assume that U is a surjective linear isometry from $L^p(\mu_1, X_1)$ onto $L^p(\mu_1, X_2)$. There exist a regular set isomorphism T from Σ_1 onto Σ_2 (defined modulo the null sets), a strongly measurable operator-valued function $t \to V(t)$, and a suitable weight function h such that for $F \in L^p(\mu_1, X_1)$,*

$$UF(t) = V(t)(h(t)TF(t)) \text{ a.e.},$$

where T is the surjective isometry induced by the set isomorphism of the same name, and $V(t)$ is a surjective isometry from X_1 onto X_2 for almost all t.

PROOF. We will give the proof for the case of finite measure spaces. By Theorem 8.2.8 there is a regular set isomorphism T of Σ_1 onto Σ_2 such that

$$U \circ \chi_A = \chi_{TA} \circ U.$$

Consider the isometry $S = U \circ \Psi^{-1}$, where Ψ denotes the isometry from $L^p(\mu_1, X_1)$ to $L^p(\mu_2, X_1)$ given by

$$\Psi F(t) = h(t)TF(t), \tag{35}$$

where T is the isometry induced by the set transformation and $h(t)$ is a function satisfying equation (21) for $1 \leq p < \infty, p \neq 2$ and $h(t) = 1$ for all t when $p = \infty$. Thus S is an isometry from $L^p(\mu_2, X_1)$ to $L^p(\mu_2, X_2)$ and we want to find a family $V(t)$ of isometries of X_1 onto X_2 so that for $G \in L^p(\mu_2, X_1)$ we have

$$SG(t) = V(t)G(t) \text{ a.e.}$$

By Proposition 8.2.9 and Zorn's lemma, we may choose a maximal family $\{W_1^\alpha\}_{\alpha \in I}$ of pairwise orthogonal separable nonzero subspaces of X_1 such that $U(L^p(\mu_2, W_1^\alpha) = L^p(\mu_2, W_2^\alpha)$ for suitable separable subspaces W_2^α of X_2. It may be shown that these spaces are pairwise orthogonal as well. Now for each $j = 1, 2$ let

$$W_j^\infty = \overline{sp}(\cup\{W_j^\alpha : \alpha \in I\}).$$

We claim that $W_1^\infty = X_1$. Certainly it is true that $S(L^p(\mu_2, W_1^\infty)) = L^p(\mu_2, W_2^\infty)$. If W_j is the orthogonal complement of W_j^∞ for each $j = 1, 2$, and if W_j is infinite-dimensional, then by the proposition there are nonzero separable subspaces W_j^0 of W_j such that

$$S(L^p(\mu_2, W_1^0)) = L^p(\mu_2, W_2^0),$$

which contradicts the maximality of the family. (If $0 < \dim W_1 < \infty$, choose $W_1^0 = W_1$.)

It follows that the Hilbert spaces $X_j = \sum_{\alpha \in I} W_j^\alpha$ are orthogonal decompositions of separable subspaces such that $S(L^p(\mu_2, W_1^\alpha)) = L^p(\mu_2, W_2^\alpha)$ for each α. By the known theorems in the separable cases established earlier, we may choose, for each $\alpha \in I$, a family $V^\alpha(t)$ of isometries from W_1^α to W_2^α such that

$$SG(t) = V^\alpha(t)G(t) \text{ a.e.}$$

for $G \in L^p(\mu_2, W_1^\alpha)$. We define $V(t) : X_1 \to X_2$ by letting its restriction to W_1^α be $V^\alpha(t)$. Since every $G \in L^p(\mu_2, X_1)$ takes its values essentially in a countable sum of the W_1^α's, we have $SG(t) = V(t)G(t)$ a.e. Since $U = S\Psi$, the proof is complete. □

8.3. L^p Functions with Values in Banach Space

It is natural to ask whether the theorems considered in the previous section can be extended to the case of general Banach spaces, and Cambern asked this right after giving his proof of Theorem 8.2.4. In particular, does every Banach space have the strong L^p-Banach-Stone property? The answer is no, as is shown by the following example.

8.3.1. EXAMPLE. *Let (Ω, Σ, μ) be a finite measure space and suppose that T is a regular set isomorphism on Σ different from the identity. Then the operator U defined by*

$$UF(t) = (f_1(t), Tf_2(t))$$

determines an isometry from $L^p(\mu, \ell^p(2))$ to itself which does not have the form given by (32). Here, f_1, f_2 denote the coordinate functions of F.

It is straightforward to verify the assertions in the example, and in fact the example is typical in the sense that the technique establishes a necessary condition for a Banach space to satisfy the strong L^p-Banach-Stone property. Let us state that formally.

8.3.2. THEOREM. *(Greim and Sourour) Let (Ω, Σ, μ) denote a finite measure space which does not consist of a single atom and suppose X is the ℓ^p direct sum $X = X_1 \oplus_p X_2$ of two nonzero subspaces. Then there is an isometry $U : L^p(\mu, X) \to L^p(\mu, X)$ that cannot be represented in the form (32).*

The necessary condition of which we speak, then, is that the space X not be an ℓ^p direct sum of two nonzero subspaces. We want to show now that this condition, along with separability, is also a sufficient condition. In the above, and in what follows in this section, we always assume that $1 \leq p < \infty, p \neq 2$, and q satisfies $\frac{1}{p} + \frac{1}{q} = 1$.

The key to the previous characterizations of isometries on L^p spaces was the fact that they must map functions of disjoint support to functions of disjoint support. In the scalar and Hilbert space cases, this fact came from a form of Clarkson's inequality. In the Banach space case we need to take a different approach and ours will be to find the Hermitian operators and use their interaction with isometries to get what we want. First, however, let us show in general, following Sourour, how the disjoint support property leads us to a characterization of the operator. We begin with a lemma.

8.3.3. LEMMA. *Suppose that U is an invertible operator from $L^p(\mu_1, X_1)$ onto $L^p(\mu_2, X_2)$ so that both U and U^{-1} have the disjoint support property. If e, u are linearly independent elements of X_1, then $U\tilde{e}(t), U\tilde{u}(t)$ are linearly independent for almost all $t \in \Sigma_2$.*

PROOF. Suppose u, e are linearly independent in X_1. Then there exists e^* in X_1^* such that $\langle\langle e, e^*\rangle\rangle = 1$ and $\langle\langle u, e^*\rangle\rangle = 0$. For each $x \in X_1$ let $F_x = U\tilde{x}$, and given $A \in \Sigma_2$, let B_x denote the support of $U^{-1}(\chi_A F_x)$. Because of the disjoint support conditions, we must have $U(\chi_{B_x} x) = \chi_A F_x$. Note that $\tilde{e^*}$ is an element of $L^q(\mu_1, X_1^*)$, and so determines a bounded linear functional on $L^p(\mu_1, X_1)$. Regarding $(U^{-1})^*$ as the adjoint map of U^{-1} from $L^q(\mu_1, X_1^*)$ to $L^q(\mu_2, X_2^*)$, we want to show that for almost all $t \in \Sigma_2$, we have

$$\langle\langle U\tilde{u}(t), (U^{-1})^*\tilde{e^*}(t)\rangle\rangle = 0 \tag{36}$$

and

$$\langle\langle U\tilde{e}(t), (U^{-1})^*\tilde{e^*}(t)\rangle\rangle \neq 0. \tag{37}$$

Let A be any set in Σ_2 and $B = B_u$ the corresponding set in Σ_1 as defined above. Then

$$\begin{aligned}
0 &= \int_B \langle\langle u, e^*\rangle\rangle d\mu_1(t) = \int \langle\langle \chi_B \tilde{u}(t), \tilde{e^*}(t)\rangle\rangle d\mu_1(t) \\
&= \langle\langle U^{-1}U(\chi_B u), \tilde{e^*}\rangle\rangle = \langle\langle U(\chi_B u, (U^{-1})^*(\tilde{e^*})\rangle\rangle \\
&= \int \langle\langle U(\chi_B u)(t), (U^{-1})^*\tilde{e^*}(t)\rangle\rangle d\mu_2(t) \\
&= \int \langle\langle \chi_A (U\tilde{u})(t), (U^{-1})^*(\tilde{e^*}(t))\rangle\rangle d\mu_2(t) \\
&= \int_A \langle\langle U\tilde{u}(t), (U^{-1})^*(\tilde{e^*}(t))\rangle\rangle d\mu_2(t).
\end{aligned}$$

Since this holds for all such A, we conclude that (36) must hold μ_2 a.e. A similar argument with u replaced by e will establish (37). □

8.3.4. THEOREM. *(Sourour) Let U be a bounded invertible operator from $L^p(\mu_1, X_1)$ onto $L^p(\mu_2, X_2)$, where X_1, X_2 are Banach spaces with X_1 separable. If both U and U^{-1} have the disjoint support property, then there is a regular set isomorphism T from Σ_1 onto Σ_2 and a strongly measurable map V of Ω_2 into $\mathcal{L}(X_1, X_2)$ such that*

$$UF(\cdot) = A(\cdot)TF(\cdot)$$

for every $F \in L^p(\mu_1, X_1)$*. Furthermore,* U *is an isometry if and only if*

$$A(t) = h(t)V(t),$$

where $V(t)$ *is an isometry of* X_1 *onto* X_2 *for almost all* t*, and*

$$h = \left(\frac{d(\mu_1 \circ T^{-1})}{d\mu_2}\right)^{1/p}.$$

PROOF. If $x \in X_1$, and $A \in \Sigma_1$, we have

$$U\tilde{x} = U(\chi_A x) + U(\chi_{A^c} x),$$

where the supports of the two functions on the right are disjoint. On the support of $U(\chi_A x)$ we must have

$$U\tilde{x}(t) = U(\chi_A x)(t) \text{ a.e.}$$

If we define $T_x(A) = \text{supp} U(\chi_A x)$, then

$$U(\chi_A x) = \chi_{T_x A} U\tilde{x}.$$

The T_x so defined can be shown to be a regular set isomorphism on Σ_1 to Σ_2. We want to show that for any $x, y \in X_1$, we have $T_x = T_y$ modulo null sets.

To that end, let x, y be given. The statement above is clearly true if x and y are dependent. Suppose, then, that x, y are independent and let $z = x + y$. From the linearity of U and some manipulation with the equalities above, we can obtain the fact that

$$[\chi_{T_z A}(t) - \chi_{T_x A}(t)]U\tilde{x}(t) + [\chi_{T_z A}(t) - \chi_{T_y A}(t)]U\tilde{y}(t) = 0 \text{ a.e.}$$

By Lemma 8.3.3, $U\tilde{x}(t), U\tilde{y}(t)$ are linearly independent a.e., so that

$$\chi_{T_z A}(t) - \chi_{T_x A}(t) = 0 = \chi_{T_z A}(t) - \chi_{T_y A}(t)$$

almost everywhere, from which we conclude that $T_x A = T_y A$ (modulo a null set).

Let us suppose that $\{e_n\}$ is a countable, linearly independent set of norm 1 elements whose span D is dense in X_1. Let T denote the set isomorphism corresponding to e_1 as above. It is clear that for any simple function F, we have $U(\chi_A F) = \chi_{TA} UF$, and by the density we get the same statement for any $F \in L^p(\mu_1, X_1)$. Since U is surjective, T must map Σ_1 onto Σ_2 (modulo null sets). As is usual, we may consider the measure $\nu = \mu_1 \circ T^{-1}$ on σ_2 and a nonnegative μ_2-measurable function h for which

$$\int_B |h(t)|^p d\mu_2(t) = \int_B \frac{d\nu}{d\mu_2} d\mu_2(t)$$

for each $B \in \Sigma_2$. Let us assume that $F_n = U(\tilde{e_n})$, where we will treat F_n as a specific function from the equivalence class. For each $t \in \Sigma_2$ let

$$A(t)e_n = F_n(t),$$

and extend A linearly to D, so that for every $y \in D$, $A(\cdot)y = U(\tilde{y})$. Let D_0 denote the elements of D with rational coefficients. If $A \in \Sigma_1$ and $y \in D_0$, we have

$$\begin{aligned}\int_{TA} \|A(t)y\|^p d\mu_2(t) &= \int \chi_{TA}\|U(\tilde{y})(t)\|^p d\mu_2(t) \\ &= \int \|U(\chi_A y)(t)\|^p d\mu_2(t) \\ &= \|U(\chi_A y)\|^p \\ &\leq \|U\|^p \mu_1(A)\|y\|^p \\ &= \|U\|^p\|y\|^p \int_{TA} |h(t)^p d\mu_2(t).\end{aligned}$$

Because this holds for all $A \in \Sigma_1$ (so that TA runs through all of Σ_2), we may conclude that $\|A(t)y\| \leq \|U\|\|y\||h(t)|$ a.e., and by the countability of D_0, there exists a null set A_0 such that

$$\|A(t)y\| \leq \|U\|\|y\|h(t) \text{ for } y \in D_0,\ t \notin A_0. \tag{38}$$

If $y = \sum_{j=1}^n \lambda_j e_j$ and Y_n denotes the span of $\{e_1, \ldots, e_n\}$, the restriction of $A(t)$ to Y_n is a linear map between two finite-dimensional spaces, and hence is bounded, while its norm must be less than or equal to $\|T\|h(t)$ since $D_0 \cap Y_n$ is dense in Y_n. This shows that (38) holds for all $y \in D$, so that $A(t)$ can be extended to a bounded linear operator satisfying the inequality (38) for all $y \in X_1$.

To show that $t \to A(t)$ is strongly measurable means to show that $A(\cdot)x$ is Bochner measurable for each $x \in X_1$. If $x \in X_1$ is given, there is a sequence $\{x_n\}$ in D which converges to x, and it follows that $A(t)x = \lim A(t)x_n = \lim U\tilde{x_n} = U(\tilde{x})$, which is measurable.

If we now let $SF(\cdot) = A(\cdot)TF(\cdot)$, then we see that

$$\begin{aligned}\int \|SF(t)\|^p d\mu_2 &\leq \int \|A(t)\|^p\|TF(t)\|^p d\mu_2(t) \\ &\leq \|U\|^p \int (h(t))^p\|TF(t)\|^p d\mu_2 \\ &\leq \|U\|^p\|F\|^p.\end{aligned}$$

Thus S is a bounded linear operator from $L^p(\mu_1, X_1)$ to $L^p(\mu_2, X_2)$ which agrees with U on simple functions and we conclude that $S = U$.

If we assume that U is an isometry, let $V(t) = A(t)/h(t)$ for $t \in \Omega_2$. (We note that $h(t)$ cannot be zero on a set of positive measure since T is a regular set isomorphism.) If $A \in \Sigma_1$, and $F = \chi_A x$, then by the definition of h and the fact that U is an isometry, we obtain

$$\begin{aligned}\int_{TA} (h(t))^p\|V(t)x\|^p d\mu_2(t) &= \mu_1(A)\|x\|^p \\ &= \int_{TA} (h(t))^p\|x\|^p d\mu_2(t).\end{aligned}$$

Since this holds for every A, we must have $\|V(t)x\| = \|x\|$ for almost all $t \in \Omega_2$. For a countable dense set $\{y_n\}$ in X_1 we can find a null set A_0 such that $\|V(t)y_n\| = \|y_n\|$ for all $t \notin A_0$, and by the boundedness of $V(t)$, we can conclude this for all $x \in X_1$ so that $V(t)$ is an isometry for almost all t.

Since U is surjective, the functions $\tilde{x_n}$ must be in the range of U and so we must have $D \subset V(t)X_1$ for all $t \notin A_0$. Since $V(t)$ is an isometry, we conclude that $V(t)X_1 = X_2$ for almost all $t \in \Omega_2$.

On the other hand, given the conditions on V and h it is straightforward to show that $UF(t) = h(t)V(t)TF(t)$ defines an isometry. □

Our goal now is to characterize the *Hermitian* operators on vector-valued L^p spaces. For this, we will assume that the spaces under consideration are complex. The reader will recall that there are several equivalent definitions of Hermitian operator, (see Theorem 5.2.6 in Chapter 5, for example), and we are going to use the one involving the numerical range. Thus an operator H is Hermitian on a Banach space X if $[Hx, x]$ is real for each $x \in X$, where $[\cdot, \cdot]$ is a semi-inner product (s.i.p.) on X which is compatible with the norm of X. It is important that this notion does not depend on a particular compatible s.i.p. on X since the convex hull of the set $\{[Hx, x] : \|x\| = 1\}$ is the same for all determinations of the s.i.p. We also recall that the equation

$$[x, y] = \langle\langle x, y^* \rangle\rangle$$

defines a compatible s.i.p. where $y^* \in X^*$ denotes a support functional corresponding to y satisfying $\langle\langle y, y^* \rangle\rangle = \|y\|^2$ and $\|y^*\| = \|y\|$. The use of the symbol $[\cdot, \cdot]$ below will assume it is a compatible s.i.p. on X which satisfies the equation above. Lastly we remark that it is always possible to choose a homogeneous s.i.p.; that is, one that satisfies $[x, \lambda y] = \overline{\lambda}[x, y]$.

8.3.5. Lemma. *Let $F \in L^p(\mu, X)$ and suppose $\int_A [F(t), x] d\mu(t) = 0$ for every $x \in X$ and every measurable subset A of a measurable set E of positive measure. Then $F(t) = 0$ a.e. on E.*

Proof. Let E, F be given as in the statement and let $x \in X$. Then $\chi_E F \in L^p(\mu, X)$ and the hypotheses imply that

$$[\chi_E(t)F(t), x] = 0 \text{ a.e.}$$

If we suppose that x^* is a support functional corresponding to x, we have

$$\langle\langle \chi_E(t)F(t), x^* \rangle\rangle = 0 \text{ a.e.}$$

By the Bishop-Phelps subreflexivity theorem, the support functionals are dense in X^*, so that the above statement holds for all $x^* \in X^*$. Now we apply Corollary 7 on page 48 in [**109**] to conclude that $F(t) = 0$ a.e. on E. □

We note that if $[\cdot,\cdot]$ is a fixed (homogeneous) s.i.p. on X and G is a simple function in $L^p(\mu, X)$, then a support functional for G is given by

$$\langle\langle F, G^*\rangle\rangle = \int [F(t), G(t)] \frac{\|G(t)\|^{p-2}}{\|G\|^{p-2}} d\mu(t), \quad F \in L^p(\mu, X). \tag{39}$$

We must mention here that it is necessary to stick to simple functions for the above statement, since, in general, $[F(t), G(t)]$ may not be measurable for all $F, G \in L^p(\mu, X)$.

8.3.6. LEMMA. *Let F_1, F_2 be simple functions in $L^p(\mu, X)$ with disjoint supports E_1, E_2, respectively. Given that $\mathcal{A}$ is a bounded Hermitian operator on $L^p(\mu, X)$, then*

(40)

$$\int_{E_2} [(\mathcal{A}F_1)(t), F_2(t)]\|F_2\|^{p-2} d\mu(t) = \overline{\int_{E_1} [(\mathcal{A}F_2(t), F_1(t)]\|F_1(t)\|^{p-2} d\mu(t)}.$$

PROOF. The proof follows from writing out the requirement that $\langle\langle \mathcal{A}(F_1 + e^{i\theta}F_2, (F_1 + e^{i\theta}F_2)^*\rangle\rangle$ is real, and using the fact that $e^{i\theta}\alpha + e^{-i\theta}\beta + \gamma$ is real for all θ implies that $\alpha = \overline{\beta}$. □

8.3.7. LEMMA. *Let $E \in \Sigma$ with $\mu(E) > 0$ and suppose $z \in X$. If $\mathcal{A}$ is a bounded Hermitian operator on $L^p(\mu, X)$, then*

$$\mathcal{A}(\chi_E z) = \chi_E \mathcal{A}(\tilde{z}). \tag{41}$$

PROOF. Given E and z as in the statement above with $\|z\| = 1$, suppose that E_1 is a subset of $\Omega \backslash E$ with $\mu(E_1) > 0$. Let $F_1 = \chi_E z$ and $F_2 = \alpha\chi_{E_1} x$, where $\|x\| = 1$ and $\alpha > 0$. If we substitute these functions into equation (40) in Lemma 8.3.6, using $\alpha = 1$ and $\alpha \neq 1$, we ultimately obtain the equation

$$(1 - \alpha^{p-2}) \int_{E_1} [\mathcal{A}(\chi_E z)(t), x] d\mu(t) = 0$$

for every $x \in X$. From Lemma 8.3.5, we conclude that $\mathcal{A}(\chi_E z)(t) = 0$ a.e. on E_1. Hence the support of $\mathcal{A}(\chi_E z)$ is contained in E and the conclusion (41) follows from this.

□

8.3.8. THEOREM. *(Sourour) An operator $\mathcal{A}$ on $L^p(\mu, X)$, where $1 \leq p < \infty, p \neq 2$, and X is a separable complex Banach space, is Hermitian if and only if $\mathcal{A}F(\cdot) = A(\cdot)F(\cdot)$ for a strongly measurable map A of Ω into $\mathcal{L}(X)$ for which $A(t)$ is Hermitian on X for almost all t.*

PROOF. If $\mathcal{A}$ satisfies the conditions of the theorem, then it is straightforward to show that for a simple function G, using equation (39), there is a s.i.p. compatible with the norm such that $[\mathcal{A}G, G]$ is real. Since such simple functions are dense, it follows from a result of Bonsall and Duncan [**49**, p. 83] that $\mathcal{A}$ is Hermitian.

Thus, let us suppose that $\mathcal{A}$ is a bounded Hermitian operator on $L^p(\mu, X)$. Using the fact that $\mathcal{A}$ is bounded and Lemma 8.3.7, we obtain for any $E \in \Sigma$ and $z \in X$,

$$\int_E \|\mathcal{A}\tilde{z}(t)\|^p d\mu(t) \le \int_E \|\mathcal{A}\|^p \|z\|^p d\mu(t).$$

Hence, it is true that

$$\|\mathcal{A}\tilde{z}(t)\| \le \|\mathcal{A}\|\|z\| \text{ for almost all } t.$$

Given a countable dense set D in X, there is a null set E_0 such that for any pair z, y in $D \cup \{0\}$,

$$\|\mathcal{A}(\tilde{z} - \tilde{y})(t)\| \le \|\mathcal{A}\|\|z - y\| \text{ for } t \in \Omega \backslash E_0.$$

It is then straightforward to show that for each $t \in \Omega \backslash E_0$, we can define $A(t)$ on $x \in X$ by

$$A(t)x = \lim \mathcal{A}(\tilde{z_n}),$$

where $\{z_n\}$ is a sequence from D converging to x. The operator so defined is well defined and $A(\cdot)x$ is easily shown to be an element of $L^p(\mu, X)$, with the property that $A(\cdot)x = \mathcal{A}\tilde{x}(\cdot)$ in $L^p(\mu, X)$. Furthermore, it is the case that $\|A(t)\| \le \|\mathcal{A}\|$ for all $t \in \Omega \backslash E_0$. The equation

$$\mathcal{A}F(t) = A(t)F(t)$$

is readily shown to hold a.e. for simple functions F and hence can be extended to all $F \in L^p(\mu, X)$ by the density of the simple functions.

It remains to show that $A(t)$ is Hermitian on X for almost all t. If $x \in X$ is given and $E \in \Sigma$, we can show, by using (39) applied to the function $\chi_E x$, that $[A(t)x, x]$ is real for almost all t, where $[\cdot, \cdot]$ is a s.i.p. compatible with the norm of X. Now we can apply this to a countable dense subset of X and use the Bonsall-Duncan result again to conclude that $A(t)$ is Hermitian a.e. □

We are finally ready to state and prove the main result of this section.

8.3.9. THEOREM. *(Sourour) Let U be an operator on $L^p(\mu_1, X_1)$ onto $L^p(\mu_2, X_2)$, $1 \le p < \infty, p \ne 2$, and suppose that X_1, X_2 are separable complex Banach spaces and X_2 has the property that it is not the ℓ^p direct sum of two nonzero Banach spaces. Then U is a surjective isometry if and only if*

$$UF(\cdot) = h(\cdot)V(\cdot)TF(\cdot), \quad F \in L^p(\mu_1, X_1), \tag{42}$$

where T is a set isomorphism from Σ_1 onto Σ_2, V is a strongly measurable map of Ω_2 into $\mathcal{L}(X_1, X_2)$ with $V(t)$ a surjective isometry from X_1 onto X_2 for almost all $t \in \Omega_2$, and $h = (d\nu/d\mu_2)^{1/p}$ where $\nu = \mu_1 \circ T^{-1}$.

PROOF. We prove only the necessity of the conditions. If U is a surjective isometry, and $\mathcal{A}$ is the operator defined on $L^p(\mu_1, X_1)$ by $\mathcal{A}F = \chi_E F$, where $E \in \Sigma_1$, then $U\mathcal{A}U^{-1}$ is necessarily a Hermitian operator on $L^p(\mu_2, X_2)$. By Theorem 8.3.8,

$$U\mathcal{A}U^{-1}F(\cdot) = P(\cdot)F(\cdot)$$

for all $F \in L^p(\mu_2, X_2)$ where $P(t)$ is a Hermitian projection for almost all $t \in \Omega_2$. Thus,

$$U(\chi_E F)(\cdot) = P(\cdot)UF(\cdot), \quad F \in L^p(\mu_1, X_1)$$

and

$$U(\chi_{E^c} F(\cdot) = Q(\cdot)UF(\cdot), \quad F \in L^p(\mu_1, X_1),$$

where $Q(t) = I - P(t)$. Let $\mathcal{M}_1, \mathcal{M}_2$ denote the subspaces of functions in $L^p(\mu_1, X_1)$ which vanish on E^c, E, respectively, while $\mathcal{N}_1 = \{P(\cdot)F(\cdot) : F \in L^p(\mu_2, X_2)\}$ and $\mathcal{N}_2 = \{Q(\cdot)F(\cdot) : F \in L^p(\mu_2, X_2)\}$. Then $U\mathcal{M}_j = \mathcal{N}_j$ for $j = 1, 2$, and since $\|F_1 + F_2\|^p = \|F_1\|^p + \|F_2\|^p$ for $F_1 \in \mathcal{M}_1$ and $F_2 \in \mathcal{M}_2$, the same must be true for elements of $\mathcal{N}_1$ and $\mathcal{N}_2$. For $x \in X_2$ and $A \in \Sigma_2$, if we let $F_1(\cdot) = P(\cdot)\chi_A(\cdot)x$ and $F_2(\cdot) = Q(\cdot)\chi_A(\cdot)x$, then $F_j \in \mathcal{N}_j$ and

$$\int_A \|P(t)x\|^p d\mu_2(t) + \int_A \|Q(t)x\|^p d\mu_2(t) = \int_A \|x\|^p d\mu_2(t).$$

Because this holds for every such A, we are able to conclude that

$$\|x\|^p = \|P(t)x\|^p + \|Q(t)x\|^p \text{ a.e.}$$

Hence there is a null set A_0 such that the above equality holds for all x in a countable dense set and all $t \notin A_0$. This can be extended to hold for all $x \in X_2$. Hence X_2 is the ℓ^p sum of $P(t)X_2$ and $Q(t)X_2$, so that by the hypotheses on X_2, we must conclude that $P(t) = 0$ or I, that is, $P(t) = \chi_E I$ a.e. for some $E \in \Sigma_2$. Thus U maps $\mathcal{M}_1$ onto the space of functions which vanish a.e. on E_c and maps $\mathcal{M}_2$ onto the space of functions which vanish a.e. on E. Therefore, U satisfies the hypotheses of Theorem 8.3.4, which gives us (42), and completes the proof. □

We remark again that the above proof holds only for complex spaces, because a description of Hermitian operators requires complex scalars. However, Greim has shown that Theorem 8.2.8 actually holds if X_1 and X_2 are separable Banach spaces. This, together with Theorem 8.3.4, gives the result of the previous theorem without the restriction to complex scalars.

8.3.10. Theorem. *(Greim) Let X_1 and X_2 be separable nonzero Banach spaces (either real or complex) that cannot be decomposed into an ℓ^p direct sum of two nonzero subspaces. Then every isometry U from $L^p(\mu_1, X_1)$ to $L^p(\mu_1, X_1)$ has the form (42).*

The previous results have excluded the case where $p = \infty$. In that case, we have the following theorem.

8.3.11. Theorem. *(Greim) Suppose that X_1, X_2 are separable Banach spaces with trivial centralizers. Then each surjective isometry U mapping $L^\infty(\mu_1, X_1)$ onto $L^\infty(\mu_2, X_2)$ has the form*

$$UF(t) = V(t)(TF)(t), \tag{43}$$

where T is the surjective isometry induced by the set isomorphism of the same name, and V is a strongly measurable operator-valued function such that $V(t)$ is an isometry from X_1 onto X_2 for almost all t.

Proof. The existence of the set isomorphism T with the desired properties can be shown exactly as in the proof of Theorem 8.2.7. Indeed the only part of that proof that needs revision here is the argument extablishing that $V(t)$ is an isometry almost everywhere. Thus, we define $S = UT^{-1}$ and observe that for any set $A \in \Sigma_2$ and $G \in L^\infty(\mu_2, X_1)$,

$$S(\chi_A G) = \chi_A SG.$$

Since this holds for all measurable A, we conclude that

$$\|SG(t)\| = \|G(t)\| \text{ a.e.} \tag{44}$$

This last equation must hold since otherwise we should have some positive integer n and a set A of positive measure on which $\|SG(t)\| \leq \|G(t)\| - 1/n$ (or with SG and G interchanged) which would clearly violate the fact that

$$\|\chi_A G\| = \|S(\chi_A G)\| = \|\chi_A SG\|.$$

In particular, for $x \in X_1$ and $t \in \Omega_2$, we may define

$$V(t)x = S\tilde{x}(t),$$

where we choose a particular representation from the equivalence class $S\tilde{x}$. It follows that V is a strongly measurable map from Ω_2 into the bounded operators from X_1 to X_2, and from (44), we have

$$\|V(t)x\| = \|x\| \text{ a.e.} \tag{45}$$

If $\{e_n\}$ is a dense sequence in X_1, let D denote its (dense) linear span, and let D_0 denote the rational span of $\{e_n\}$. From (45), we can select a μ_2-null set A_0 such that

$$\|V(t)e_n\| = \|e_n\|$$

for all $t \notin A_0$. This can be applied to the countable collection of elements in D_0 and finally extended to all of D. Since D is dense, we can obtain

$$\|V(t)x\| = \|x\|$$

for all $x \in X_1$ and all $t \in \Omega_2 \backslash A_0$. Thus $V(t)$ is an isometry a.e., and the remainder of the proof follows as in the proof of Theorem 8.2.7. □

As a final remark, we note that the assumption of the measure spaces being finite in the results of this section can be relaxed to include the case of σ-finite measure spaces.

8.4. L^2 Functions with Values in a Banach Space

In the previous sections we have always excluded the case when $p = 2$. In general, Hilbert spaces have too many isometries for precise description, and the techniques of those sections break down. However, Lin has shown that something can be said for describing Hermitian operators and isometries for the spaces $L^2(\mu, X)$ where X is a Banach space with suitable properties. The approach to characterizing the isometries will involve the description of the Hermitian operators, and so our spaces will be assumed to be complex. First we will describe the Hermitian operators on $X_1 \oplus_2 X_2$, where X_1, X_2 are Banach spaces. We begin with a lemma.

8.4.1. Lemma. *(Lin) If* $H = \left[\begin{smallmatrix} 0 & H_1 \\ H_2 & 0 \end{smallmatrix}\right]$ *is a Hermitian operator on* $X_1 \oplus_2 X_2$*, where* X_1, X_2 *are complex Banach spaces, then* $Y_1 = \overline{H_1 X_2}$ *and* $Y_2 = \overline{H_2 X_1}$ *are isometrically isomorphic to Hilbert spaces, and there exist two subspaces* Z_1, Z_2 *of* X_1, X_2*, respectively, such that*

$$X_1 = Z_1 \oplus_2 Y_1 \ \textit{ and } \ X_2 = Z_2 \oplus_2 Y_2. \tag{46}$$

Proof. If $[\cdot,\cdot]_j$ denotes a homogeneous s.i.p. compatible with the norm on X_j, $j = 1, 2$, then

$$[(x_1, x_2), (y_1, y_2)] = [x_1, y_1]_1 + [x_2, y_2]_2$$

defines a s.i.p. compatible with the norm of $X_1 \oplus_2 X_2$. Furthermore, if

$$H = \begin{bmatrix} 0 & H_1 \\ H_2 & 0 \end{bmatrix}$$

is a Hermitian operator on $X_1 \oplus_2 X_2$, where H_1 is an operator from X_2 into X_1 and H_2 is from X_1 to X_2, then

$$[H_1 x_2, x_1]_1 + [H_2 x_1, x_2]_2 \in \mathbb{R},$$

for any $x_1 \in X_1$ and $x_2 \in X_2$. Upon replacing x_2 by ix_2 in the above we get that

$$i\{[H_1 x_2, x_1]_1 - [H_2 x_1, x_2]_2\} \in \mathbb{R}.$$

From this we may conclude that for all $x_1 \in X_1, x_2 \in X_2$, it is true that

$$[H_1 x_2, x_1]_1 = \overline{[H_2 x_1, x_2]_2}.$$

Hence, we see that $H_1 = 0$ if and only if $H_2 = 0$. Now if $x_1, y_1 \in X_1$ and $x_2 \in X_2$, we have

$$\begin{aligned} [H_1 x_2, x_1 + y_1]_1 &= \overline{[H_2(x_1 + y_1), x_2]_2} \\ &= \overline{[H_2 x_1, x_2]_2} + \overline{[H_2 y_1, x_2]_2} \\ &= [H_1 x_2, x_1]_1 + [H_1 x_2, y_1]_1. \end{aligned} \tag{47}$$

The restriction of $[\cdot,\cdot]_1$ to $H_1 X_2$ is therefore a homogeneous s.i.p. compatible with the norm such that

$$[x, y + z]_1 = [x, y]_1 + [x, z]_1 \ \text{ for } \ x, y, z \in H_1 X_2. \tag{48}$$

A homogeneous s.i.p. satisfying (48) is an inner product, and therefore, $Y_1 = \overline{H_1 X_2}$ is a Hilbert space. A similar argument implies that $Y_2 = \overline{H_2 X_1}$ is also a Hilbert space. Note that for any $y \in Y_1$, $x \in X_1$, and s.i.p. $[\cdot,\cdot]$ it follows from (47) that

$$[y, \alpha x + \beta y] = [y, \alpha x] + [y, \beta y] \text{ for all } \alpha, \beta \in \mathbb{C}. \tag{49}$$

It remains to show the existence of subspaces Z_1, Z_2 satisfying (46). Recall that for elements x, y in a Banach space, we say that x is orthogonal to y, written $x \perp y$, if $\|x\| \le \|x + \alpha y\|$ for all $\alpha \in \mathbb{C}$ (see Section 4 of Chapter 1). If $[y, x] = 0$ for a compatible s.i.p., then $x \perp y$, and if $x \perp y$, then there is a compatible (homogeneous) s.i.p. such that $[y, x] = 0$. First we show that if $x \in X_1$ and $y \in Y_1$ are as defined above, with $x \perp y$, then the span of x, y is isometric to $\ell^2(2)$.

To this end, suppose x, y are as given just above and assume that each has norm 1. We will establish a series of facts.

(i) Since $x \perp y$, we may choose a compatible s.i.p. such that $[y, x] = 0$. If $\|\alpha x + \beta y\| = 1$, then

$$|\beta| = |[y, \beta y]| = |[u, \alpha x] + [y, \beta y]| = |[y, \alpha x + \beta y]| \le 1. \tag{50}$$

Hence for any α, we have

$$1 = \left\| \frac{\alpha x}{\|\alpha x + y\|} + \frac{y}{\|\alpha x + y\|} \right\|,$$

and so by (50), $1 \le \|\alpha x + y\|$, which, since $\|y\| = 1$, means that $y \perp x$. From this we may conclude that there is a compatible (homogeneous) s.i.p. with both $[x, y] = 0$ and $[y, x] = 0$.

(ii) If $Y = sp\{x, y\}$, we claim that the norm of Y is smooth on

$$Y \backslash (\{\beta y : \beta \in \mathbb{C}\} \cup \{\alpha x : \alpha \in \mathbb{C}\}).$$

Suppose $\|\alpha x + \beta y\| = 1$ with $|\alpha| \ne 1 \ne |\beta|$. If the norm of Y is not smooth at $\alpha x + \beta y$, then there are two homogeneous s.i.p.s $[\cdot,\cdot]$ and $[\cdot,\cdot]'$ which satisfy

$$0 = [x, y] = [y, x] = [y, x]' = [x, y]'$$

and are distinct on Y. Now,

$$[\alpha x + \beta y, \alpha x + \beta y] = 1 = [\alpha x + \beta y, \alpha x + \beta y]'$$

and by (49),

$$[y, \alpha x + \beta y] = [y, \beta y] = \overline{\beta} = [y, \alpha x + \beta y]'.$$

It is readily seen from these facts that

$$[z, \alpha x + \beta y] = [z, \alpha x + \beta y]'$$

for all $z \in Y$, which contradicts the distinctness of the two s.i.p.s.

(iii) Given any $0 \le \alpha \le 1$, there is a unique $\beta \ge 0$ such that $\|\alpha x + \beta y\| = 1$. This follows from the continuity and convexity of the norm as a function of β on $[0, \infty)$. The same holds also for $0 \ge \alpha \ge -1$.

(iv) For $0 \leq \alpha \leq 1$ (respectively, $0 \geq \alpha \geq 0$), let $f(\alpha)$ be the unique nonnegative real number such that $\|\alpha x + f(\alpha)y\| = 1$. By the smoothness of the norm shown above, the function $f(\alpha)$ is differentiable on $0 < \alpha < 1$ (respectively, $-1 < \alpha < 0$) and there exists c such that

$$[\cdot, \alpha x + f(\alpha)y] = c\{[\cdot, -f'(\alpha)x] + [\cdot, y]\}.$$

Because $[\alpha x + f(\alpha)y, \alpha x + f(\alpha)y] = 1$ and $[y, \alpha x + f(\alpha)y = f(\alpha)$, we see that

$$c = \frac{1}{-\alpha f'(\alpha) + f(\alpha)}$$

while

$$\frac{1}{-\alpha f'(\alpha) + f(\alpha)} = f(a) \text{ and } -\alpha f'(\alpha)f(\alpha) = 1 - f^2(\alpha).$$

The derivative of $(1 - f^2(\alpha))/\alpha^2$ is zero so there is a constant k such that $1 - f^2(\alpha) = k\alpha^2$. Since $f(1) = 0 = f(-1)$, we conclude that $k = 1$ and

$$f(\alpha) = \sqrt{1 - \alpha^2}.$$

The conclusion is that $\mathrm{sp}\{x, y\}$ is a Hilbert space.

Since Y_1 is a Hilbert space and reflexive, it is true that for every $x \in X_1$ there is some $y \in Y_1$ such that

$$\|x - y\| = \inf_{w \in Y_1} \|x - w\|,$$

that is, Y_1 is *proximinal* in X_1.

Let

$$Z_1 = \left\{ z \in X_1 : \|z\| = \inf_{y \in Y_1} \|z - y\| \right\}.$$

If $z \in Z_1$ and $y \in Y_1$, we have $\|z - \alpha y\| \geq \|z\|$ and from the previous arguments we get that $\{z, y\}$ is an orthogonal basis for $sp\{z, y\}$ so that

$$[y, z] = 0 = [z, y].$$

If $v \in Z_1$, then (again using (49)) we obtain

$$[y, z + v] = [y, z] + [y, v] = 0.$$

It follows from this that

$$\|z + v\| \leq \|z + v + \alpha y\| \text{ for all } \alpha \in \mathbb{C},$$

and since this holds for every $y \in Y_1$ we are able to infer that $z + v \in Z_1$, and therefore, Z_1 is a subspace of X_1. Furthermore, given $x \in X_1$, there exists, since Y_1 is proximinal, $y \in Y_1$ such that

$$\|x - y\| = \inf_{w \in Y_1} \|x - w\|.$$

Hence, $z = x - y \in Z_1$ and because $sp\{z, y\}$ is a Hilbert space, we have

$$x = z + y \in Z_1 + Y_1 \text{ and } \|x\| = \sqrt{\|z\|^2 + \|y\|^2}.$$

Similarly, we can find a subspace Z_2 so that $X_2 = Z_2 \oplus_2 Y_2$. This completes the proof. □

It is of no difficulty to verify that if

$$H = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}$$

is a Hermitian operator on $X_1 \oplus_2 X_2$, then

$$\begin{bmatrix} H_1 & 0 \\ 0 & H_4 \end{bmatrix}$$

is Hermitian and H_1 and H_4 are Hermitian on X_1, X_2, respectively. Furthermore,

$$\begin{bmatrix} 0 & H_2 \\ H_3 & 0 \end{bmatrix}$$

is Hermitian. The next theorem now follows from Lemma 8.4.1.

8.4.2. THEOREM. *(Lin) Suppose that X_1, X_2 are Banach spaces such that neither may be written in the form $Z \oplus_2 \mathbb{C}$ for a subspace Z. If*

$$H = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}$$

is a Hermitian operator on $X_1 \oplus_2 X_2$, then $H_2 = H_3 = 0$, and H_1, H_4 are Hermitian operators on X_1, X_2, respectively.

Let us say that a complex Banach space X has property (L) if there is a subspace Y of X such that $X = Y \oplus_2 \mathbb{C}$. Note that if X fails to have property (L), then the dimension of X is at least 2.

8.4.3. LEMMA. *(Lin) If X is a complex Banach space which fails to have property (L), then $L^2(\mu, X)$ also fails to have property (L).*

PROOF. If $L^2(\mu, X)$ does have property (L), there exists $F \in L^2(\mu, X)$ such that if F, G are linearly independent, then $sp\{F, G\}$ is isometric with $\ell^2(2)$. Suppose A is the support of G and $F|_A \neq 0$. Then

$$\begin{aligned}
&\int_A \|F(t) + G(t)\|^2 d\mu + \int_{\Omega\setminus A} \|F(t)\|^2 d\mu \\
&= \|F + G\|^2 = \|F\|^2 + \|G\|^2 \\
&= \int_A \|F(t)\|^2 d\mu + \int_A \|G(t)\|^2 d\mu + \int_{\Omega\setminus A} \|F(t)\|^2 d\mu.
\end{aligned}$$

Thus, $sp\{F|_A, G\}$ is isometrically isomorphic to $\ell^2(2)$.

Since F is a nonzero element of $L^2(\mu, X)$, there is some $x \in X$ with $x \neq 0$ such that for any $\epsilon > 0$,

$$\mu\{t \in \Omega : \|F(t) - x\| < \epsilon\} > 0.$$

Let $A_\epsilon = \{t \in \Omega : \|F(t) - x\| < \epsilon\}$, and let $y \in X$ such that $[y, x] = 0$ and $\|y\| = \|x\|$. Let T be the mapping from $sp\{x, y\}$ onto $sp\{F|_{A_\epsilon}, \chi_{A_\epsilon}\}$ so that

$$T(x) = F|_{A_\epsilon} \text{ and } Ty = \chi_{A_\epsilon} y.$$

Then

$$\|T\| \cdot \|T^{-1}\| \leq \frac{1}{(1-\epsilon)^2}.$$

Therefore $sp\{x,y\}$ is nearly isometric to $\ell^2(2)$, and so is isometric to it [**18**, p. 242]. Now if $sp\{x,y\}$ is isometric to $\ell^2(2)$ for any y such that $[y,x]=0$, then we must have $X = Y \oplus_2 sp\{x\}$, which implies that X has property (L). □

Now we will show that Hermitian operators and isometries can be described as in the previous section, provided the spaces do not have property (L).

8.4.4. THEOREM. *(Lin) Suppose that X is a separable Banach space without property (L) and (Ω, Σ, μ) is σ-finite. An operator H is Hermitian on $L^2(\mu, X)$ if and only if*

$$H(F(\cdot) = A(\cdot)F(\cdot), \tag{51}$$

where $t \to A(t)$ is a Hermitian-valued strongly measurable map from Ω into $\mathcal{L}(X)$.

PROOF. Let $A \in \Sigma$ have positive measure. Then we may write

$$L^2(\mu, X) = L^2(\mu|_A, X) \oplus_2 L^2(\mu|_{A^c}, X).$$

By Lemma 8.4.3, the fact that X fails property (L) implies that neither space on the right in the equation above has property (L), and so by Theorem 8.4.2 we have

$$H(L^2(\mu|_A, X) \subset L^2(\mu|_A, X)$$

and

$$H(L^2(\mu|_{A^c}, X) \subset L^2(\mu|_{A^c}, X).$$

This is just what is needed to use the argument given in the proof of Theorem 8.3.8 to establish the conclusion of the present theorem.

□

8.4.5. THEOREM. *(Lin) Let $(\Omega_j, \Sigma_j, \mu_j)$, for $j = 1, 2$, be σ-finite measure spaces, and suppose U is an operator from $L^2(\mu_1, X)$ onto $L^2(\mu_2, Y)$, where X, Y are separable Banach spaces which have dimension greater than 1 and have nontrivial L^2 structure. Then U is a surjective isometry if and only if*

$$UF(\cdot) = V(\cdot)h(\cdot)TF(\cdot), \quad F \in L^2(\mu_1, X), \tag{52}$$

where T is a set isomorphism from Σ_1 onto Σ_2, V is a strongly measurable map of Ω_2 into $\mathcal{L}(X, Y)$ with $V(t)$ a surjective isometry from X onto Y for almost all $t \in \Omega_2$, and $h = (d\nu/d\mu_2)^{1/2}$ where $\nu = \mu_1 \circ T^{-1}$.

PROOF. Note that the hypotheses on X and Y imply that they fail to have property (L). If we assume that U is a surjective isometry and $A \in \Sigma_1$

with $\mu_1(A) > 0$, then the operator H defined by $HF = \chi_A F$ is Hermitian and so UHU^{-1} is Hermitian on $L^2(\mu_2, Y)$. By Theorem 8.4.4, we have

$$UHU^{-1}G(\cdot) = P(\cdot)G(\cdot)$$

for all $G \in L^2(\mu_2, Y)$ where $P(t)$ is a Hermitian projection for almost all $t \in \Omega_2$. The remainder of the proof can be given just as in the proof of Theorem 8.3.9. □

8.5. Notes and Remarks

It is not surprising, considering his many contributions to understanding vector-valued function spaces, that Cambern seems to be the first person to consider the problem of describing the isometries on Banach space-valued L^p spaces. His 1974 paper [**65**] on the isometries of $L^p(\mu, X)$, where X is a separable Hilbert space, opened the door to this study. The list of questions mentioned in the beginning were first displayed by Greim and Jamison, [**157**] and the first reference to the L^p-Banach-Stone property was by Greim [**154**] (in analogy with the language coined by Cambern [**66**] and modified by Behrends [**27**] for continuous function spaces).

The description of the $L^p(X)$ spaces and the definition of Bochner measurability is taken from the book by Diestel and Uhl [**109**]. An important fact that we used without identification is given in the following theorem stated and proved in [**109**, p. 42].

8.5.1. Theorem. *(Pettis's measurability theorem) A function $F : \Omega \to X$ is Bochner measurable if and only only if*

(i) *F is μ-essentially separably valued, and*
(ii) *F is weakly μ-measurable.*

It is also pointed out by Diestel and Uhl that the simple functions are dense in $L^p(X)$ for $1 \leq p < \infty$ and that the countably valued functions are dense in $L^\infty(X)$.

***L^p* Functions with Values in Hilbert Space.** The first four numbered results in this section are due to Cambern and may be found in [**65**]. The proofs are essentially his, although we have used the notion of Bochner measurable where Cambern used weak measurability. Omitted details may be found in Cambern's paper. Theorem 8.2.4 is also proved in [**130**] using Hermitian operators and the method of Lumer. (See Chapter 5.) This approach leads to a generalization of Lamperti's theorem [**229**], whereas the proof of Theorem 8.2.4 makes use of Lamperti's theorem. We note the triviality that in the proof of Theorem 8.2.4 the separable Hilbert space X could be replaced by a pair X_1, X_2 of spaces without requiring any substantive change at all.

Cambern followed in 1981 with a treatment of the $p = \infty$ case [**74**] for separable Hilbert spaces. We have followed Greim's approach, which was also generally applied to the case where X_1, X_2 are Banach spaces. The concept of *lifting*, which appears in the proofs, was discussed by Greim [**152**] with references to the work of A. and C. Tulcea [**188**]. It is an interesting way to formally handle the use of equivalence classes, and we thought it worth showing how it enters the arguments. Our proof of Theorem 8.2.5 is basically taken from [**152**] where the given proof, according to Greim, is a simplified version of one given in [**153**]. The statements about the coincidence of the strong and norm topologies on $Z(X)$ can be found in [**27**]. Lemma 8.2.6 is taken directly from [**74**, Lemma 3]. The proof of Theorem 8.2.7 basically follows the proof of Theorem 2 in [**153**], although it uses ideas of Cambern [**74**] and Sourour [**354**].

An isometry U satisfying the equation displayed in the statement of Theorem 8.2.8 is said to be "reduced by a set isomorphism." The proof of this theorem is due to Greim [**154**] and makes use of integral modules and the L^p structure developed by Behrends and his cohorts [**27**] and [**32**]. It provides us with the necessary set isomorphism. Although the theorem as given in [**154**] does not state it for $p = \infty$, it is true in that case. The argument at the beginning of the proof of Theorem 8.2.7 could also be used in this case. The proofs of Proposition 8.2.9 and Theorem 8.2.11 are in [**157**].

As we have remarked before, the results in this section hold for σ-finite measure spaces. In fact, Cengiz [**88**] has shown that the result for $L^p(\mu, X)$ $(1 \leq p < \infty,\ p \neq 2)$, where X is a Hilbert space, actually holds for arbitrary measure spaces. He uses the fact that for such p, there is a *perfect* measure space for which the two L^p spaces are isometric. Then he proves the theorem for perfect measure spaces. We will not discuss perfect measure spaces here, but the reader may consult [**88**] and its references.

L^p **Functions with Values in Banach Space.** The material in this section is largely drawn from the 1978 paper of Sourour [**354**]. Example 8.3.1 is the one given in [**130**], but was also given independently by Sourour in [**354**] who stated Theorem 8.3.2 for separable measure spaces. Sourour noted that his theorem was valid for measure spaces which admit a set isomorphism different from the identity. Greim [**155**] showed that this must always be true of a measure space not consisting of a single atom.

Sourour's techniques in the proof of his Theorem 3.1 and Corollary 3.2 in [**354**] have served a number of authors working at establishing the form of an isometry on a vector- valued L^p space. Our version of these results is given in Theorem 8.3.4. Sourour stated the theorem for a bounded operator on $L^p(\mu, X)$, but the validity of the proof in the nonsurjective case has been questioned [**225**]; indeed, a counterexample is given by Koldobsky in that paper. Here is the example. Suppose X is isometric to a subspace of $L^p([0,1])$ so that $L^p([0,1], X)$ is isometric to a subspace of the space $L^p([0,1] \times [0,1])$ of scalar functions on the unit square. Since this space is isometric to $L^p([0,1])$, there exists an isometry V from $L^p([0,1], X)$ to $L^p([0,1])$. Let e be an element of X with norm 1, and define U from $L^p([0,1], X)$ into itself by $UF(t) = VF(t)e$ for all $t \in [0,1]$ and $F \in L^p([0,1], X)$. Then U is an isometry that can not be represented in the form (42).

The characterization of Hermitian operators on $L^p(\mu, X)$ was given by Sourour [**354**], and also for X a Hilbert space, in [**130**]. Elements of the proof of Theorem 8.3.8 have been taken from both of those articles. Lemma 8.3.5 is our attempt to provide a proof of the statement given by Sourour in his proof [**354**, p. 282]. The proof of Lemma 8.3.6 uses ideas first used by Tam [**365**]. The expression (39) for the s.i.p. and the remarks about it are given by Sourour [**354**, pp. 281-282].

We should say something about our reference to the Bishop-Phelps subreflexivity theorem. A Banach space X is said to be *subreflexive* if the set of norm-attaining functionals in X^* is dense in X^*. Bishop and Phelps proved in 1961 [**47**] that every Banach space is subreflexive. It follows that the support functionals are therefore dense, which is what we need. There is an excellent discussion of these matters in Section 2.11 of Megginson's book [**281**].

The theorem of Greim which extends Sourour's theorem to the real case, Theorem 8.3.10, is proved in [**155**]. This paper gives a description of the L^p *structure* of $L^p(\mu, X)$. By the L^p structure of a Banach space X is meant the set of all L^p projections, that is, those projections P satisfying

$$\|x\|^p = \|Px\|^p + \|x - Px\|^p$$

for all $x \in X$. This structure is said to be *trivial* if it consists only of zero and the identity. Thus the sufficient (and necessary) condition of the main theorem could be expressed as saying that X has trivial L^p-structure.

Greim [**156**] has shown that any Banach space (separable or not) has the L^1-Banach-Stone property; that is, there is a regular set isomorphism which reduces the isometry. Koldobsky [**225**] shows that the isometries of $L^p(\mu, L^k(\nu))$ have the canonical form when $p \neq 1, k \neq 2, p \neq k$, and $k \notin (p, 2)$ for $p < 2$. This paper has a number of interesting results and references, and includes a study of the structure of isometries on subspaces of $L^p(\mu, L^k(\nu))$. This latter work is thus related to what is discussed in Section 3 of Chapter 3. Koldobsky also notes that in his dissertation, he showed that $C(K)$ spaces also have the strong L^p-Banach-Stone property for $p > 1$. Among other things, Koldobsky shows in [**226**] that if X is one of the spaces $C(K)$ or L^1 and Y is an arbitrary Banach space, then X is isometric to a subspace of $L^p(\mu, Y)$, $p > 1$, only if X is isometric to a subspace of Y. Furthermore, he showed that an isometry U from X into $L^p(\mu, Y)$ is of the form $Ux(\cdot) = h(\cdot)V(\cdot)x$, where h is a measurable function and V is a strongly measurable isometry-valued map. These results do not hold for $p = 1$. For example, the two-dimensional space $\ell^\infty(2)$ is isometric to a subspace of $L^1([0, 1])$, and so also to a subspace of $L^1([0, 1], Y)$ for any Banach space Y. If the above statements were true for $p = 1$, then we would have to conclude that $\ell^\infty(2)$ is isometric to a subspace of any Banach space Y [**226**].

Lin [**257**] has extended Koldobsky's results to the case where Y is replaced by the vector- valued Köthe function spaces $E(Y)$ when E is strictly convex.

Theorem 8.3.11 was proved by Greim in [**153**], although we have added a few details to show how to get the isometries $V(t)$. The proof also owes much to the techniques developed by Sourour in [**354**]. In [**154**], Greim shows that Banach dual spaces (separable or not) with trivial centralizers have the strong L^∞-Banach-Stone property, except that the operators $V(t) : X_1 \to X_2$ can only be shown to be of norm 1 a.e. In this same paper, it is shown that $C(K)$ spaces, even though they do not, in general, have trivial centralizers, do have the Banach-Stone property for L^∞ spaces in that an isometry leads to an isomorphism of the underlying measure spaces.

L^2 **Functions with Values in a Banach Space.** The results of this section are due to Lin [**253**]. In the proofs of Theorems 8.4.4 and 8.4.5 we have referred back to the proofs given in Section 3, rather than to the theorems of Sourour [**354**, 3.1, 3.2, 4.2, 5.2] as did Lin [**253**]. We should note that Lin stated Theorems 8.4.4 and 8.4.5 in much more generality, in that he assumed the spaces $L^2(\mu_1, X)$ and $L^2(\mu_2, Y)$ were themselves ℓ^2 direct sums of spaces $L^2(\mu_j, X_j)$ and $L^2(\mu'_j, Y_j)$, respectively.

Other Vector-Valued Function Spaces. Although we have not devoted a section to this topic in the main text, we do want to briefly mention some results on isometries for vector-valued H^p spaces and Orlicz spaces.

It was Cambern in 1972 [**64**] who extended the de Leeuw, Rudin, Wermer, Nagasawa theorem [**106**], [**289**] (see Chapter 4, and in particular Theorem 4.2.2) to the space $H^\infty(X)$ where X is a finite-dimensional (complex) Hilbert space. This was extended by Cambern a few years later [**69**] to the case where X is a finite dimensional Banach space satisfying certain conditions. The first result for $H^\infty(X)$ where X is infinite-dimensional came in about 1988 by Lin [**254**], who established the theorem for the case in which X is a uniformly convex and uniformly smooth Banach space. Shortly after that, Cambern and Jarosz [**81**] proved the following theorem, which contained all of the earlier results.

8.5.2. THEOREM. *(Cambern and Jarosz) Let X be a Banach space with Mult $(X) = \mathbb{C}$, and let T be an isometry of $H^\infty(D, X)$ onto itself, where D is the open unit disk. Then T is of the form*

$$(TF)(z) = \mathcal{T}F(\tau(z)), \quad F \in H^\infty(D, E), \ z \in D,$$

where $\mathcal{T}$ is a constant isometry of X onto X and τ is a conformal map of the disk onto itself.

By Mult(X), of course, is meant the family of multipliers of X which is defined back in Chapter 7.

About the same time, Cambern and Jarosz [**79**] took up the case when $p = 1$ and showed that if X is a finite-dimensional complex Hilbert space, and T is a surjective isometry of $H^1(D, X)$ onto itself, then

$$TF(z) = UF(\tau(z))\tau'(z)$$

for $F \in H^1(D, X)$, $z \in D$, τ a conformal map of the disk, and U a unitary operator on X. Two years later Lin [**255**] extended this result to obtain the theorem given below.

8.5.3. THEOREM. *(Lin) Let $1 \leq p < \infty$, $p \neq 2$, and suppose X is any complex Hilbert space. If $T : H^p(D, X) \to H^p(D, X)$ is a surjective isometry, then there exist a unitary operator U on X and a conformal map φ from the disk onto itself such that*

$$(TF)(z) = U(F \circ \varphi(z)) \cdot (d\varphi/dz)^{1/p}(z)$$

for $F \in H^p(D, X), z \in D$.

Hornor and Jamison [**182**] have characterized the isometries on the space $S^p(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, generalizing a theorem of Novinger and Oberlin [**293**] (see Chapter 4, Theorem 4.5.1).

In another direction, we mention a result of Jamison and Loomis [**194**] that extends the Lamperti-type theorems of the earlier sections to a vector-valued Orlicz space. For this we suppose that (Ω, Σ, μ) denotes a nonatomic measure space with $\mu(\Omega) = 1$ and X is a separable complex Hilbert space. Let φ denote a continuous, strictly increasing convex function on $[0, \infty)$ with $\varphi(0) = 0, \varphi(1) = 1$, and $\lim_{t\to\infty} \varphi(t) = \infty$. Suppose also that φ and its complementary function satisfy the Δ_2 condition. (See Section 2 of Chapter 5.) The space $L^\varphi(\mu, X)$ will denote the set of measurable functions F from Ω to X for which

$$\int \varphi\left(\frac{\|F(t)\|}{\lambda}\right) d\mu < \infty$$

for some $\lambda > 0$. Then $L^\varphi(\mu, X)$ is a Banach space with the Luxemburg norm given by

$$\|F\| = \inf\left\{\int \varphi\left(\frac{\|F(t)\|}{\lambda}\right) d\mu \leq 1\right\}.$$

Here is the theorem.

8.5.4. THEOREM. *(Jamison and Loomis) If $\varphi(t) \neq ct^2$ for some c, then a linear operator U is a surjective isometry of $L^\varphi(\mu, X)$ if and only if there exists a regular set isomorphism T of the measure space and a $\mathcal{L}(X)$-valued function $V(\cdot)$ such that*

$$UF(\cdot) = h(\cdot)V(\cdot)TF(\cdot),$$

where $F \in L^\varphi(\mu, X)$, $h = \varphi^{-1}(d\nu/d\mu)$, $\nu = \mu \circ T^{-1}$, and $V(t)$ is unitary for almost all $t \in \Omega$.

The proof is similar to earlier proofs in which the Hermitian operators are characterized and then used to describe the isometries.

Carrying this generalization a bit further, Randriantoanina [**308**] has extended the above description to the isometries on the space she denotes by

$X(H)$. Here, H is a Hilbert space and X is a (real or complex) rearrangement-invariant function space on Ω (where $\Omega = [0, 1]$ or $\Omega \subset \mathbb{N}$) whose norm is not proportional to the L^2 norm.

Again we end with a few more references: [**40**], [**78**], [**95**], [**171**], [**223**], [**224**], and [**304**].

CHAPTER 9

Orthogonal Decompositions

9.1. Introduction

In the previous chapters we have considered function spaces where the values of the functions are in Hilbert spaces or Banach spaces. In this chapter we continue this focus, but here we will take the underlying function spaces to be discrete Banach function spaces. Thus it will be natural to consider our spaces to be sums of Banach spaces with certain properties. Such sums can be realized as special cases of what M.M. Day called *substitution* spaces. These are defined as follows. Suppose S is an index set and E is a Banach space of scalar functions with the property that if $y \in E$ and $|x(s)| \leq |y(s)|$ for all $s \in S$, then $x \in E$ and $\|x\| \leq \|y\|$. For each s suppose that X_s is a Banach space and let $X = E((X_s)_{s \in S})$ denote the functions x such that $x(s) \in X_s$ for each $s \in S$ and the function $|x| \in E$ where $|x|(s) = \|x(s)\|$. Then X is a Banach space with the norm $\|x\| = \| \, |x| \, \|_E$. If each X_s is equal to the same space Y, we write $X = E(Y)$. The condition put on E above means that it must have what is usually called an *absolute* norm; that is, the norm of a function depends only on its absolute values.

In the case where the space E is a sequence space, the corresponding substitution space is often called the E *direct sum* of the X_s spaces. Hence by $\ell^p(c_0)$, for example, we would mean the ℓ^p direct sum of a sequence of c_0 spaces. Even the general substitution spaces are often regarded as "direct sum" spaces (written as $(\sum_S \oplus X_s)_E$), and we shall see that surjective isometries on such spaces usually preserve the direct sum structure. The main thrust of this chapter is to provide evidence for the previous statement.

We begin our investigations with a look at the paper of Schneider and Turner which is focused on the finite-dimensional case. Although certainly not the first paper that treats sequence spaces of sums of spaces, this treatment foreshadows much of the work on orthogonal decompositions that we want to relate in this chapter. In this section we will show how the space $\mathbb{C}^n$ furnished with an absolute norm can be written as the direct sum of finite-dimensional Hilbert spaces. The surjective isometries can be described in the form $T = UP$, where U is an isometry of the form $\exp(iH)$, where H is Hermitian and P is a permutation matrix. In fact, our results are more general than that and apply to spaces which are infinite-dimensional complex spaces with one-unconditional bases. In Section 3 we will give an exposition of Kalton and Wood's work on general orthonormal systems which extends the previous results to not necessarily countable systems. The methods there are elegant and have had many applications.

The work we have mentioned above is devoted entirely to complex spaces. In the fourth section, we follow Rosenthal's treatment of the real case by considering Banach spaces that are isometric to the direct sum of Hilbert spaces via a one-unconditional basis (called *functional Hilbertian sums.*) Here many of the results of Section 3 are obtained for real spaces. Finally, in Section 5, we will examine some more recent results of Li and Randrianantoanina concerning direct sums of sequence spaces. The goal is to indicate a way to decompose a sequence space (real or complex) with an absolute norm into a direct sum of simpler spaces, which are not necessarily Hilbert spaces, and to use the decomposition to obtain characterizations of isometries.

As usual, we close the chapter with a section of notes and remarks. The following definition will be of use throughout the chapter.

9.1.1. DEFINITION. *A collection $\{x_\alpha : \alpha \in \Gamma\}$ in a Banach space X is said to be a* one-unconditional *or* hyperorthogonal *basis for X if for every $x \in X$ there exist scalars $\{a_\alpha : \alpha \in \Gamma\}$ such that*

$$x = \sum_{\alpha \in \Gamma} a_\alpha x_\alpha$$

and

$$\left\| \sum a_\alpha x_\alpha \right\| = \left\| \sum |a_\alpha| x_\alpha \right\|.$$

If $\|x_\alpha\| = 1$ for each α, we say the basis is normalized.

By the sum here we mean, of course, the convergence of the partial sums over the net of finite subsets of Γ directed by inclusion.

9.2. Sequence Space Decompositions

In this section we will let X denote a complex Banach sequence space which contains a normalized one-unconditional basis. Hence X may be regarded as a sequence space with a norm that depends only on the absolute values of the coordinates and in which the norm is *standardized*; that is, the standard unit basis vectors or coordinate vectors $e_j = (\delta_{jk})$ each have norm 1. We will show that such spaces can be written as direct sums of Hilbert spaces. In our work in this section we are influenced and inspired by the work of Schneider and Turner on finite-dimensional spaces, but we will treat our spaces as being infinite-dimensional. Because we will be dealing with several possible norms on a space, in this chapter we will commonly use Greek letters like μ, ν, and ρ for norms on X and μ^*, ν^* for the corresponding norms on the dual space. By $\| \cdot \|_2$, we will mean the usual ℓ^2 norm. For elements x of X we will sometimes write $x(j)$ for the jth coordinate and $x = (x(j))$ for the sequence itself.

At the beginning here we will give some definitions and results which hold for general sequence spaces with absolute norm, but not necessarily requiring that the vectors e_j form a Schauder basis.

9.2.1. DEFINITION. *Let X be a sequence space with absolute, standardized norm ν. We will say that two coordinates j, k are equivalent with respect to ν if $\nu(x) = \nu(y)$ whenever*

$$|x(j)|^2 + |x(k)|^2 = |y(j)|^2 + |y(k)|^2$$

and $|x(m)| = |y(m)|$ for all $m \neq j, k$. If j and k are equivalent, we will write

$$j \sim k.$$

Note: The reader may naturally ask whether the 2 is significant in the above definition, or whether one could use p or something else. Section 5 will treat the more general situation. Here we want to follow the historical development and to produce a sum of Hilbert spaces, which is itself of natural and practical interest.

9.2.2. LEMMA. *Let $N = \{1, 2, \ldots, n\}$ or $\mathbb{N}$ depending on whether X is n-dimensional or infinite-dimensional. Then the relation $\sim$ defines an equivalence relation on N.*

PROOF. We prove only the transitive condition. Suppose that $j \sim k$ and $k \sim r$. Let $x, y \in X$ satisfy the condition for j, r as given in Definition 9.2.1. Let $\tilde{x}$ be the sequence with

$$\tilde{x}(j) = \tilde{x}(r) = 0, \quad \tilde{x}(k) = \sqrt{|x(j)|^2 + |x(k)|^2 + |x(r)|^2}$$

and $\tilde{x}(m) = x(m)$ for all $m \neq j, k, r$. Let $\hat{x}$ have $\hat{x}(k) = \sqrt{|x(k)|^2 + |x(r)|^2}$, $\hat{x}(r) = 0$, and $\hat{x}(m) = x(m)$ for all $m \neq k, r$. Then we must have $\nu(\hat{x}) = \nu(\tilde{x})$ since $j \sim k$, and $\nu(x) = \nu(\hat{x})$ since $k \sim r$. Thus $\nu(\tilde{x}) = \nu(x)$. If we let $\tilde{y}$ be the corresponding element to y, it is clear that $\nu(\tilde{x}) = \nu(\tilde{y})$, from which we conclude that $\nu(x) = \nu(y)$. □

We want to insert a lemma here that establishes the equivalence of three ways to express that a norm is absolute.

9.2.3. LEMMA. *Let E be a sequence space with norm ν and for which the coordinate vectors form a normalized one-unconditional basis. The following are equivalent.*

(i) *If $x, y \in E$ and $|x| = (|x(j)|) = (|y(j)|) = |y|$, then $\nu(x) = \nu(y)$.*

(ii) *If $\{\beta_j\}$ is a sequence of scalars with $|\beta_j| \leq 1$ for each j, then $(\beta_j x(j)) \in E$ and*

$$\nu((\beta_j x(j))) \leq \nu(x).$$

(iii) *If $y \in E$ and $|x(j)| \leq |y(j)|$ for each j, then $x \in E$ and $\nu(x) \leq \nu(y)$.*

If the condition about the basis is removed, the norm conditions still hold, given that all indicated sequences are in E.

PROOF. (i) ⇒ (ii). Given $|\beta_j| \leq 1$, there exist α_{1j} and α_{2j} of absolute value 1 such that

$$\beta_j = \frac{\alpha_{1j} + \alpha_{2j}}{2}.$$

Hence

$$\nu(\sum_{j=n+1}^{n+k} \beta_j x(j) e_j) \leq \frac{1}{2}\nu(\sum_{j=n+1}^{n+k} \alpha_{1j} x(j) e_j) + \frac{1}{2}\nu(\sum_{j=n+1}^{n+k} \alpha_{2j} x(j) e_j)$$
$$= \frac{1}{2}\nu(\sum_{j=n+1}^{n+k} x(j) e_j) + \frac{1}{2}\nu(\sum_{j=n+1}^{n+k} x(j) e_j).$$

It follows that the sequence of partial sums for $\sum_{j\geq 1} \beta_j x(j))$ is Cauchy, so the sequence $((\beta_j x(j))$ is in E, and the indicated norm inequality must hold.
(ii) ⇒ (iii). If $|x(j)| \leq |y(j)|$ for each j, we can define $\beta_j = \dfrac{x(j)}{y(j)}$ so that $x = (\beta_j y(j))$ and the result is immediate. (Here, of course, we define $\beta_j = 0$ if $y(j) = 0$.)
(iii) ⇒ (i). This is obvious.

For the final remark, we note that it is unnecessary and unhelpful in this case to worry about the Cauchy condition. The norm inequalities in each case follow directly. □

In the remainder of this section, we will let $N_1, N_2, \cdots$ denote the equivalence classes determined by the equivalence relation given by 9.2.2. We will treat these sets as if there are infinitely many of them and that each is infinite. The adjustments in the finite cases should be obvious. The elements of N_j will be denoted by $p_{j1}, p_{j2}, \cdots$ in increasing order.

9.2.4. THEOREM. *Let (X, ν) be a sequence space for which the unit vectors $\{e_j\}$ form a normalized one-unconditional basis. Then there exists a sequence of subspaces $\{X_j\}$ of X, each isometric with ℓ^2 (of appropriate dimension), and a sequence space (E, μ) with no equivalent coordinates, whose coordinate vectors form a normalized one-unconditional basis such that $X = E((X_j))$.*

PROOF. For each positive integer j let X_j denote the closed linear span of the basis vectors $\{e_{p_{jk}} : p_{jk} \in N_j\}$. If $x = \sum x(k)e_k$, let $x_j = \sum_k x(p_{jk})e_{p_{jk}}$. Then $x_j \in X_j$ and $x = \sum_j x_j$. For suppose $\epsilon > 0$ is given. Choose a positive integer M so that $\nu(\sum_{k\geq n} x(k)e_k) < \epsilon$ for $n \geq M$. Next choose M_0 so large that $\{1, 2, \ldots, M\} \subset \cup_{j<M_0} N_j$. Then the monotone properties of ν require that

$$\nu(\sum_{j\geq k} x_j) \leq \nu(\sum_{j\geq k} x(j)e_j) < \epsilon$$

for all $k \geq M_0$. This shows that X can be decomposed as a sum of the X_j spaces, sometimes called a Schauder decomposition.

Given $x \in X$, let $x^1 = x - x_1$. Since $p_{12} \sim p_{11}$ we must have

$$\nu(x) = \nu(x^1 + (|x(p_{11})|^2 + |x(p_{12})|^2)^{1/2} e_{p_{11}} + \sum_{k \geq 3} x(p_{1k}) e_{p_{1k}}).$$

It follows by induction that for every $n, k \geq 3$,

$$\nu(x) = \nu(x^1 + (\sum_{k=1}^{n} |x(p_{1k}|^2)^{1/2} e_{p_{11}} + \sum_{k>n} x(p_{1k}) e_{p_{1k}}). \tag{53}$$

Again by the monotone property of ν, we get

$$\nu(x) \geq \nu((\sum_{k=1}^{n} |x(p_{1k})|^2)^{1/2} e_{p_{11}})$$

for each n. Hence, we must have $\sum_{k \geq 1} |x(p_{1k})|^2 < \infty$ and $\nu(x_1) = \|x_1\|_2$. Thus X_1 is isometric with ℓ^2, and since $\nu(\sum_{k \geq n} x(p_{1k}) e_{p_{1k}}) \to 0$ as $n \to \infty$, we conclude from (53) that

$$\nu(x) = \nu(x^1 + \|x_1\|_2 e_{p_{11}}). \tag{54}$$

In a similar way we can argue by induction that for any positive integer r, and $x^r = x^{r-1} - x_r$,

$$\nu(x) = \nu(x^r + \sum_{k=1}^{r} \|x_k\|_2 e_{p_{k1}}). \tag{55}$$

If $y = \sum_{k=r}^{r+s} x_k$, then $y^{r+s} = 0$, and from the previous equation we get $\nu(y) = \nu(\sum_{k=1}^{r+s} \|y_k\|_2 e_{p_{k1}}$ so that

$$\nu(\sum_{k=r}^{r+s} \|x_k\|_2 e_{p_{k1}}) = \nu(\sum_{k=r}^{r+s} x_k).$$

From this we infer that the series $\sum_{k \geq 1} \|x_k\|_2 e_{p_{k1}}$ must converge to some element of X. Furthermore, (55) and the monotonicity of ν yields

$$\nu(x) = \nu(\sum_{j \geq 1} \|x_j\|_2 e_{p_{j1}}) \tag{56}$$

since $\nu(x^r) \to 0$ as $r \to \infty$.

Let E denote the space of all scalar sequences $\alpha = (\alpha(j))$ for which $\sum_j \alpha(j) e_{p_{j1}} \in X$, and define a norm μ on E by $\mu(\alpha) = \nu(\sum_j \alpha(j) e_{p_{j1}})$. Then E is a sequence space whose coordinate vectors form a normalized one-unconditional basis. Furthermore, given $x \in X$, the sequence $(\|x_j\|_2) \in E$, and $\nu(x) = \mu((\|x_j\|_2))$ by (56).

Finally, let us suppose that $j \sim k$ relative to μ. Let $r \in N_j, s \in N_k$ be given. Assume further that $x, y \in X$ with $|x(r)|^2 + |x(s)|^2 = |y(r)|^2 + |y(s)|^2$ and $|x(l)| = |y(l)|$ for all $l \neq r, s$. It is easy to see that $\|x_j\|_2^2 + \|x_k\|_2^2 = \|y_j\|_2^2 + \|y_k\|_2^2$ and $\|x_t\|_2 = \|y_t\|_2$ for $t \neq j, k$. Hence, $\mu((\|x_j\|_2)) = \mu(\|y_j\|_2))$ from which it follows that $\nu(x) = \nu(y)$. The conclusion is that $r \sim s$ with respect to ν, which implies that N_j and N_k are the same equivalence class. Therefore, $j = k$ and the proof is complete.

□

It is clear from the above proof that the sequence space X as given in the theorem is isometric to Hilbert space if and only if any two coordinates are equivalent.

To describe the isometries on our space X we intend to use what we have called Lumer's method and make use of the notion of semi-inner product, which we have done already in the previous chapter. We used the language *support functional* for the functional x^* corresponding to x such that $\|x^*\| = \|x\|$ and $\langle\langle x, x^*\rangle\rangle = \|x\|^2$. The mapping $x \to x^*$ is often called a duality map. We recall that

$$[x, y] = \langle\langle x, y^*\rangle\rangle \tag{57}$$

defines a s.i.p. which is compatible with the norm.

9.2.5. Definition. *A s.i.p.* $[\cdot,\cdot]$ *on a sequence space* (E, μ) *is said to be* d-admissible *if there exists a sequence of nonnegative real-valued functions* $\{a_j\}$ *defined on* E *such that:*

(i) $a_k(x) = a_k(y)$ *for each* k *if* $|x| = |y|$;
(ii) $a_k(\lambda x) = a_k(x)$ *for every scalar* λ, *and every* k;
(iii) *For every pair of positive integers* $j \neq k$ *there exist scalars* $x(j), x(k)$ *such that* $a_k(x) = a_j(x) \neq 0$ *for* $x = x(j)e_j + x(k)e_k$;
(iv) *For every pair of positive integers* $j \neq k$ *there exists* $x \in E$ *with* $a_j(x) \neq a_k(x)$ *and both nonzero;*
(v) *For all* $x, y \in E$,

$$[x, y] = \sum x(j)\overline{y(j)}a_j(y).$$

An example of a sequence space with a d-admissible s.i.p. is ℓ^p for $1 \leq p < \infty (p \neq 2)$. In this case, we have

$$[x, y] = \sum x(j)\overline{y(j)} \left(\frac{|y(j)|}{\|y\|_p}\right)^{p-2}.$$

If E is a sequence space whose coordinate vectors form a normalized one-unconditional basis, we will say that E is *admissible.*

9.2.6. Lemma. *The dual space* E^* *for an admissible sequence space* (E, μ) *can itself be regarded as a sequence space with absolute norm* μ^*.

Proof. Clearly, if $f \in E^*$, we must have $f(x) = \sum x(j)f(e_j)$ converges for each $x \in E$, so that f can be identified with the sequence $(f(e_j))$. On the other hand, any sequence $\beta = (\beta(j))$ for which the series $\sum x(j)\beta(j)$ converges for each $x \in E$ determines a continuous linear functional on E. Thus, for a given $f \in E^*$, since $\sum x(j)|f(e_j)|$ converges for each $x \in E$, we have $|f| \in E^*$. Note that

$$|f(x)| \leq \sum |x(j)||f(e_j)| \leq \mu^*(|f|)\mu(x)$$

so that $\mu^*(f) \leq \mu^*(|f|)$. Given $x \in E$, for each j let $y(j) = x(j)|f(e_j)|/f(e_j)$ if $f(e_j) \neq 0$ and otherwise let $y(j) = 0$. Then

$$|f|(x) = f(|y|) \leq \mu^*(f)\mu(y) \leq \mu^*(f)\mu(x).$$

Hence, we get $\mu^*(|f|) \leq \mu^*(f)$. □

In general, the coordinate vectors need not be a basis for the dual space, but if the basis for E is shrinking, then E^* will be admissible as well.

We will find it convenient in the sequel to refer to an element y of E^* to be *dual to x with respect to μ* if

$$\langle\langle x, y\rangle\rangle = \mu(x)\mu^*(y). \tag{58}$$

We observe here that if y is dual to x with respect to μ, then

$$\Re\langle\langle z - x/\mu(x), y\rangle\rangle = 0$$

so that y is normal to the support hyperplane

$$H = \{z : \Re\langle\langle z, y\rangle\rangle = \mu^*(y)\}$$

when all of the coordinates are real.

Our immediate goal now is to show that an admissible sequence space without equivalent coordinates admits a d-admissible s.i.p. To prepare for this, we need to establish a special geometric lemma which shows the existence of an element which has a dual that is a multiple of itself.

9.2.7. LEMMA. *(Schneider and Turner) Let ρ be a standardized absolute norm on $\mathbb{C}^2$. Then there exists $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2_+$ with $\alpha_1 > 0$, $\alpha_2 > 0$ and $\rho(\alpha) = 1$ such that*

$$1 = \langle\langle \alpha, \langle\alpha, \alpha\rangle^{-1}\alpha\rangle\rangle = \rho^*(\langle\alpha, \alpha\rangle^{-1}\alpha)\rho(\alpha). \tag{59}$$

PROOF. Let K denote the closed unit ball for the norm ρ. Then the intersection K_r of K with $\mathbb{R}^2$ is a convex body in $\mathbb{R}^2$. Let Γ be the part of the boundary of K_r that lies in the first quadrant. Then $(1, 0)$ and $(0, 1)$ both lie in Γ. We claim there is a point P on Γ such that both coordinates of P are positive and the line l perpendicular to OP (where O is the origin) is a support line to K_r. Let $(r(\theta)\cos\theta, r(\theta)\sin\theta)$ represent the points on Γ, so that $r(\theta)$ is a continuous function on $[0, \pi/2]$ with minimum m and maximum M satisfying $0 < m \leq 1 \leq M$.

Case 1. $1 = m = M$. In this case, Γ is a quarter circle and any point on Γ will satisfy our claim.

Case 2. $1 < M$ with $r(\theta_0) = M$. Then the part of K_r in the first quadrant is contained in a circle with center at O and radius M. The point P corresponding to $r(\theta_0)$ is the desired point.

Case 3. $m < 1$ with $r(\theta_1) = m$. Let P be the corresponding point. If the perpendicular l to OP is not a support line to K_r, suppose l' is a support line at P which is not perpendicular to OP. The slope of l' is necessarily negative, and the perpendicular to l' from O meets l' in a point Q in the first quadrant. Now the length $|OQ|$ is less than $|OP|$, and since Q is either in the exterior of K_r or on Γ, we must have a point R on OQ which is on Γ and $|OR| < m$. This is a contradiction.

Let the coordinates of the point P from above be denoted by $\alpha = (\alpha_1, \alpha_2)$. The support line l to K_r at P is given by

$$l = \{\beta : \langle \beta, \alpha \rangle = \langle \alpha, \alpha, \rangle\}.$$

It follows that for all $\beta \in \mathbb{R}^2_+$ we have

$$\langle \beta, \alpha \rangle \leq \langle \alpha, \alpha \rangle \rho(\beta).$$

Since ρ is absolute, it is also true that

$$|\langle \beta, \alpha \rangle| \leq \langle \alpha, \alpha \rangle \rho(\beta)$$

for all $\beta \in \mathbb{C}^2$. It is now easy to see that $\rho^*(\alpha) = \langle \alpha, \alpha \rangle$ and (59) follows since $\rho(\alpha) = 1$.

□

For a given pair of integers $j < k$, we will let E' denote the subspace of E spanned by e_j, e_k and by E'' the closed linear span of the remaining coordinate vectors. Then for an admissible space E we can write $x = x' \oplus x''$ for any $x \in E$ where $x' \in E'$ and $x'' \in E''$. Also, for a fixed vector $\alpha = (\alpha_1, \alpha_2)$ as guaranteed by Lemma 9.2.7, where the norm ρ in the lemma is determined by μ restricted to elements of E', let

(60) $$z' = \alpha_1 e_j + \alpha_2 e_k.$$

There is such a $z' \in E'$, of course, for every pair j, k, but we will not indicate that in the notation.

9.2.8. Lemma. *Let (E, μ) be an admissible sequence space which has no equivalent coordinates with respect to μ. Then E has a d-admissible s.i.p.*

Proof. Let φ be a duality map from E to E^* which satisfies $\varphi(\lambda x) = \overline{\lambda}\varphi(x)$ for all $x \in E$. Here we will denote by $\varphi_j(x)$ the jth coordinate of the sequence determined by $\varphi(x)$. We will show that the s.i.p. defined by (57) is d-admissible. We may also assume that φ satisfies the property that

(61) $$\varphi(|x|) = |\varphi(x)|,$$

where $|\varphi(x)| = (|\varphi_j(x)|)$. For if ψ is a duality map without that property, define $\varphi(x) = (\varphi_j(x))$ by

$$\varphi_j(x) = \begin{cases} \frac{\overline{x(j)}}{|x(j)|}\psi_j(|x|) & \text{if } x(j) \neq 0; \\ 0 & \text{if } x(j) = 0. \end{cases}$$

Then φ will define a duality map satisfying (61) as desired.

In fact, we want to put one more requirement on our duality map. We will assume that for $x = \lambda z'$ where z' is as described by (60), we have

$$\varphi(x) = \frac{\overline{x(j)}}{\langle \alpha, \alpha \rangle} e_j + \frac{\overline{x(k)}}{\langle \alpha, \alpha \rangle} e_k.$$

If this is not the case for our given duality map ψ, define φ by letting $\varphi(x)$ be defined as above for those x in the span of z', and let $\varphi(x) = \psi(x)$ for all other x. Then φ can be shown to have all of the desired properties.

We have $\langle\langle x, \varphi(y) \rangle\rangle = \sum x(j)\varphi_j(y)$ and by properties of the duality map, we obtain

$$\begin{aligned} \mu(y)^2 = \langle\langle y, \varphi(y) \rangle\rangle &= \sum y(j)\varphi_j(y) \\ &\leq \sum |\varphi_j(y)||y(j)| \\ &\leq \mu^*(|\varphi(y)|)\mu(|y|) = \mu(y)^2. \end{aligned}$$

Thus, it is true that $y(j)\varphi_j(y) = |y(j)||\varphi_j(y)|$ for each j, so that $\varphi_j(y)/\overline{y(j)}$ is real and, in fact, nonnegative whenever $y(j) \neq 0$. Hence, we define

$$a_k(y) = \begin{cases} \frac{\varphi_k(y)}{\overline{y(k)}}, & \text{if } y(k) \neq 0; \\ 0, & \text{if } y(k) = 0. \end{cases}$$

Then

$$[x, y] = \sum x(j)\varphi_j(y) = \sum x(j)\overline{y(j)} a_j(y)$$

and (v) is satisfied.

The property of φ given by (61) is just what is needed to establish

$$a_k(|x|) = \frac{\varphi_k(|x|)}{|x(k)|} = \frac{|\varphi_k(x)|}{|x(k)|} = |a_k(x)| = a_k(x).$$

Hence, for $|x| = |y|$, we have

$$a_k(x) = a_k(|x|) = a_k(|y|) = a_k(y)$$

and (i) is proved.

Property (ii) follows immediately since $\varphi(\lambda x) = \overline{\lambda}\varphi(x)$.

Given positive integers $j < k$, let z' be the vector defined by (60). It is clear that $z'(j) > 0, z'(k) > 0$ and the fact that $a_j(z') = a_k(z')$ follows from the property (59). This establishes (iii).

It remains now to show that (iv) is satisfied. For this, let us suppose that for a certain pair j, k of positive integers it is the case that $a_j(x) = a_k(x)$ for all x such that $x(j) \neq 0$ and $x(k) \neq 0$. For such an x, let $u = |x|/\mu(x)$ and write $u = u' \oplus u''$ (as indicated in the paragraph just above the statement of the lemma). We can also write $\varphi(u) = \varphi'(u) \oplus \varphi''(u)$ and remember that all of the coordinates of these elements are nonnegative. Because $a_j(x) = a_k(x)$, and by properties of a_j and a_k, we conclude that

$$u(j)\varphi_k(u) = u(k)\varphi_j(u).$$

As a result, we have

$$\varphi'(u) = du' \text{ where } d = \frac{\varphi_j(u)}{u(j)} = \frac{\varphi_k(u)}{u(k)} \geq 0. \tag{62}$$

Now $\mu(u) = 1$ and so let us suppose that $\mu(u'') < 1$. From this we must have $\langle\langle u'', \varphi''(u)\rangle\rangle < 1$ so that

$$\langle\langle u', \varphi'(u)\rangle\rangle = 1 - \langle\langle u'', \varphi''(u)\rangle\rangle > 0. \tag{63}$$

This inequality guarantees that $\varphi'(u) \neq 0$, so that $d > 0$, where d is as given in (62). We define

$$c^{-1} = 1 - \langle\langle u'', \varphi''(u)\rangle\rangle, \tag{64}$$

and so $c > 0$ also. Let $K_{u''}$ denote the set of all elements v' of E' with real coordinates and such that $\mu(v' \oplus u'') \leq 1$. Then $K_{u''}$ may be thought of as a convex body in $\mathbb{R}^2$. Let ρ be the norm determined by $K_{u''}$. Note that $u' \in K_{u''}$ with $\rho(u') = 1$ and it is straightforward to show that $\rho^*(\varphi'(u)) = c^{-1}$. From (63), (64), and (62) we obtain

$$1 = \langle\langle u', cdu'\rangle\rangle = \rho(u')\rho^*(cdu').$$

Thus, a multiple of u' is dual to u' and by the remarks made prior to the statement of Lemma 9.2.7, we can conclude that u' is orthogonal to the support hyperplane to $K_{u''}$ at the point u'. In fact, this is true for any w' on the boundary of $K_{u''}$. Hence, $K_{u''}$ is a circular disk centered at the origin (thought of as lying in $\mathbb{R}^2$).

All of this argument above was based on the assumption that $\mu(u'') < 1$. Suppose, then, that $\mu(u'') = 1$. Let $v', w' \in E'$, both having real coordinates, and suppose $v' \in K_{u''}$. Let us suppose further that $\|v'\|_2 = \|w'\|_2$. Let ϵ be given with $0 < \epsilon < 1$. Since $\mu(v' \oplus u'') \leq 1$, and μ is absolute, we must also have $\mu(v' \oplus (1-\epsilon)u'') \leq 1$. Hence, $v' \in K_{(1-\epsilon)u''}$ which is circular, so $w' \in K_{(1-\epsilon)u''}$ as well (because $\|v'\|_2 = \|w'\|_2$). From this we can show that

$$\mu(w' \oplus u'') \leq 1 + \epsilon,$$

and since this holds for all such ϵ, we have to conclude that $w' \in K_{u''}$. Wherefore, $K_{u''}$ must be circular.

We finish the argument now by showing that $j \sim k$, which would contradict the hypotheses that there are no equivalent coordinates. Thus, suppose x, y are given with $\|x'\|_2 = \|y'\|_2$ and $|x''| = |y''|$. If we let $u = |x|/\mu(x)$ and $v = |y|/\mu(x)$, then we have $\|u'\|_2 = \|v'\|_2$, $|u''| = |v''|$, and $\mu(u) = 1$. By our previous arguments, $K_{u''}$ is circular and contains u'. Therefore, $v' \in K_{u''}$ so that $\mu(v' \oplus u'') \leq 1$ and because $|v' \oplus u''| = |v' \oplus v''|$, we get $\mu(v' \oplus v'') = \mu(v) \leq 1$. This leads to the conclusion that

$$\mu(|y|) \leq \mu(|x|).$$

Reversing the roles of x and y gives the reverse inequality, and ultimately the fact that $\mu(x) = \mu(y)$.

□

Let us agree to call an admissible sequence space *pure* if it has no equivalent coordinates.

At long last, we are ready to find a description of the isometries on admissible sequence spaces. Lumer's method requires that we first describe the Hermitian operators, as we did in the previous chapter. Recall that a bounded operator H on a Banach space X is *Hermitian* if there is a s.i.p. $[\cdot, \cdot]$ compatible with the norm of X, so that $[Hx, x] \in \mathbb{R}$ for all $x \in X$. We will state our theorem for a space $X = E((X_j))$ which is a direct E-sum of Hilbert spaces. We will represent bounded operators T on such spaces by an operator matrix $[T_{jk}]$, where T_{jk} is the operator from X_k to X_j defined by $T_{jk} = P_j T Q_k$. Here, P_j is the projection of X onto X_j defined by $P_j((x_m)) = x_j$ and Q_k is the injection operator from X_k into X so that $Q_k(x_k) = (y_m)$ where $y_k = x_k$ and $y_m = 0$ for $m \neq k$.

9.2.9. THEOREM. *Suppose that $\{X_j\}$ is a sequence of Hilbert spaces and E is an admissible sequence space which is pure. Let $H = [H_{jk}]$ be a Hermitian operator on the space $X = E((X_j))$. Then $H_{jk} = 0$ for $j \neq k$, and H_{jj} is Hermitian on the Hilbert space X_j for each j. Furthermore, $\|H\| = \sup_j \|H_{jj}\|$. Conversely, any such operator is necessarily Hermitian on X.*

PROOF. We will let ν be the norm on X and μ the norm on E. Let $\{a_j\}$ be the functions as given by Definition 9.2.5 for the s.i.p. of the space (E, μ). This is possible by Lemma 9.2.8. Given $x = \sum x_j \in X$, let us abuse the notation slightly by letting $a_j(x)$ denote the value of a_j at the element $(\|x_j\|_2)$ of E. Then it is easy to see that for $x, y \in X$,

$$[x, y] = \sum_j \langle x_j, y_j \rangle a_j(y) \tag{65}$$

defines a s.i.p. compatible with the norm ν. To simplify notation, we will write x_j to indicate both an element of X_j and the element $Q_j x_j$ of X.

To begin with, note that for the operator H and any j we have

$$[Hx_j, x_j] = \langle H_{jj} x_j, x_j \rangle a_j(x_j)$$

is real for all $x_j \in X_j$, so that H_{jj} is a Hermitian operator on the Hilbert space X_j. Next suppose $x = x_j \oplus x_k$ and write out $[Hx, x]$. From what we have already seen about H_{jj} and H_{kk} we must conclude that

$$\langle H_{jk} x_k, x_j \rangle a_j(x) + \langle H_{kj} x_j, x_k \rangle a_k(x) \in \mathbb{R} \tag{66}$$

for all x_j, x_k and so

$$\begin{aligned} &\langle H_{jk} x_k, x_j \rangle a_j(x) + \langle H_{kj} x_j, x_k \rangle a_k(x) \\ &= \overline{H_{jk} x_k, x_j \rangle} a_j(x) + \overline{\langle H_{kj} x_j, x_k \rangle} a_k(x). \end{aligned} \tag{67}$$

Let $x' = x_j \oplus i x_k$. Then $a_l(x') = a_l(x)$ for all l since $|x'| = |x|$. Upon replacing x_k by $i x_k$ in (67), adding the result to that of (67), and dividing out the i, we are left with

$$\langle H_{jk} x_k, x_j \rangle a_j(x) = \overline{\langle H_{kj} x_j, x_k \rangle} a_k(x) \tag{68}$$

for all possible choices of x_j, x_k. By property (iii) of Definition 9.2.5 there exist x'_j, x'_k so that $a_k(x') = a_j(x')$ for $x' = x'_j \oplus x'_k$. Letting x_j, x_k be arbitrary members of X_j, X_k, respectively, we take

$$x = \|x'_j\|_2 \frac{x_j}{\|x_j\|_2} + \|x'_k\|_2 \frac{x_k}{\|x_k\|_2}$$

so that $a_j(x) = a_j(x') = a_k(x') = a_x(x)$. Upon substitution of this into (68) we obtain

$$\langle H_{jk} x_k, x_j \rangle = \overline{\langle H_{kj} x_j, x_k \rangle} \tag{69}$$

for all x_k, x_j.

For any x we can write, using (69),

$$\begin{aligned}[Hx, x] = &\sum_{1 \le p < q} (\langle H_{pq} x_q, x_p \rangle a_p(x) + \overline{\langle H_{pq} x_q, x_p \rangle} a_q(x)) \\ &+ \sum_p \langle H_{pp} x_p, x_p \rangle a_p(x)\end{aligned} \tag{70}$$

is real. For a given j and x, let x' be the same as x except the jth summand is replaced by $e^{i\theta} x_j$ for some θ between 0 and 2π. Substituting x' for x in (70) yields

$$\begin{aligned} e^{i\theta} &\left(\sum_{1 \le p < j} \langle H_{pj} x_j, x_p \rangle a_p(x) + \sum_{q > j} \overline{\langle H_{jq} x_q, x_j \rangle} a_q(x) \right) \\ &+ e^{-i\theta} \left(\sum_{1 \le p < j} \overline{\langle H_{pj} x_j, x_p \rangle} a_j(x) + \sum_{q > j} \langle H_{jq} x_q, x_j \rangle a_j(x) \right) \\ &+ \gamma \end{aligned}$$

is real for all choices of θ where γ is real and independent of θ. It is straightforward to show that if an expression $e^{i\theta}\alpha + e^{-i\theta}\beta + \gamma$ is real for all choices of θ, then $\alpha = \overline{\beta}$. Applying this idea to the expression above gives

$$\sum_{1 \le p < j} \langle H_{pj} x_j, x_p \rangle (a_p(x) - a_j(x)) + \sum_{q > j} \overline{\langle H_{jq} x_q, x_j \rangle} (a_q(x) - a_j(x)) = 0. \tag{71}$$

By properties of the a_j's, we could replace each term in (71) by its absolute value without changing $a_p(x)$ for any x. Thus,

$$\langle H_{jk} x_k, x_j \rangle (a_j(x) - a_k(x)) = 0 \tag{72}$$

for every choice of x.

Suppose $H_{jk} \neq 0$ for some $j \neq k$. Hence there exist y_j, y_k so that $\langle H_{jk} y_k, y_j \rangle \neq 0$. Given any x with $x_j \neq 0, x_k \neq 0$, let $z = \sum z_j$, where

$$z_j = \frac{\|x_j\|_2}{\|y_j\|} y_j, \quad z_k = \frac{\|x_k\|}{\|y_k\|} y_k$$

and $z_l = x_l$ for $l \neq j, k$. Now $\langle H_{jk}z_k, z_j\rangle \neq 0$ and we conclude from (72) that

$$a_j(x) = a_j(z) = a_k(z) = a_k(x).$$

The assumption that $H_{jk} \neq 0$ leads to $a_j(x) = a_k(x)$ for all x, which contradicts (iv) of the properties of a d-admissible s.i.p.

The converse follows easily by calculating $[Hx, x]$. □

A space satisfying the hypotheses of the previous theorem will be said to have a *pure Hilbert space decomposition.*

In Lemma 9.2.8, we showed that a pure space must have a d-admissible s.i.p. We can use the previous theorem to obtain a converse of that result.

9.2.10. COROLLARY. *If E is an admissible space with d-admissible s.i.p., then E has no equivalent coordinates.*

PROOF. By Theorem 9.2.9, the only Hermitian operators on E are diagonal operators with real eigenvalues. If $j \sim k$, then $H = e_k^* \otimes e_j + e_j^* \otimes e_k$ is Hermitian (which can be verified by showing that e^{itH} is an isometry for each t). However, H is not diagonal, which is a contradiction. (By $x^* \otimes x$ we mean, as usual, the rank 1 operator whose value at y is $x^*(y)x = \langle\langle y, x^*\rangle\rangle x$.) □

9.2.11. COROLLARY. *(Schneider and Turner) Let X be an admissible sequence space (or a space with normalized one-unconditional basis). Then X can be expressed as a direct sum space $E((X_j))$ where E is pure and the X_j's are Hilbert spaces. A bounded operator H is norm Hermitian on X if and only if its operator matrix $[H_{jk}]$ is diagonal with H_{jj} self-adjoint as a Hilbert space operator for each j. Moreover, if $[h_{jk}]$ is the matrix representation for H with respect to the coordinate basis vectors, then H is norm Hermitian if and only if*

(i) $h_{jk} = \overline{h_{kj}}$ *for* $j \sim k$, *and*

(ii) $h_{jk} = 0$ *for* $j \nsim k$.

PROOF. By Theorem 9.2.4, X has the given decomposition, where the sequence space E is admissible and has no equivalent coordinates. Hence by Lemma 9.2.8, E has a d-admissible s.i.p. Therefore, X is a space as given in the hypotheses of Theorem 9.2.9 and the stated result follows. The last part comes from the fact that a given equivalence class of coordinates determines one of the self-adjoint Hilbert space operators H_{jj} whose matrix representation is equal to its conjugate transpose.

□

Corollary 9.2.11 is a more general form of Theorem 5.2.13 in Chapter 5. Note that it says that a Hermitian operator on c_0 or ℓ^p for $1 \leq p < \infty, p \neq 2$ must be a real diagonal operator. That is, the matrix representation for such an operator with respect to the coordinate vectors is diagonal with real eigenvalues. Let us now describe the isometries. We first introduce some terminology which will make the result easier to state. Given a sequence space with absolute norm μ, the *symmetry class* S_μ of the norm relative to the coordinate vectors is the class of all permutations π of the positive integers for which $\mu((x(j)) = \mu((x(\pi(j)))$.

9.2.12. THEOREM. *Suppose that $\{X_j\}$ is a sequence of Hilbert spaces and E is a pure admissible complex sequence space. For T to be an isometry of $X = E((X_j))$ onto itself, it is necessary and sufficient that there exist $\pi \in S_\mu$ such that*

(i) *for each j, $TX_{\pi(j)} = X_j$ and*
(ii) *for $x = \sum x_j$, $(Tx)_j = T_{j\pi(j)}x_{\pi(j)}$ for each j and $T_{j\pi(j)}$ is a unitary map of $X_{\pi(j)}$ onto X_j.*

By $(Tx)_k$ we mean the kth "coordinate" or kth summand of Tx.

PROOF. Suppose ν denotes the norm on X and μ the norm on E. If the conditions (i) and (ii) are satisfied, then

$$\|(Tx)_j\|_2 = \|T_{j\pi(j)}x_{\pi(j)}\|_2 = \|x_{\pi(j)}\|_2 \text{ for each } j.$$

Therefore,

$$\nu(Tx) = \mu((\|(Tx)_j\|_2)) = \mu((\|x_{\pi(j)}\|_2)) = \mu((\|x_j\|_2)) = \nu(x)$$

since $\pi \in S_\mu$.

For the necessity of the conditions, let T be an isometry of X onto X. We will use the fact that UHU^{-1} is Hermitian whenever H is Hermitian. Let j be given and let H_{jj} be any Hermitian on the Hilbert space X_j. If $T = (T_{kp}), T^{-1} = (B_{kp})$ and H is the operator given by the operator matrix which has H_{jj} in the correponding position on the diagonal with all other entries equal to the zero operator, we must have

$$THT^{-1} = (T_{kj}H_{jj}B_{jp}).$$

By Theorem 9.2.9, we must have $T_{kj}H_{jj}B_{jp} = 0$ for all k, p with $k \neq p$. Because H_{jj} can be any Hermitian operator on X_j, it follows that for any bounded operator W on X_j, we have $T_{kj}WB_{jp} = 0$. Hence, if $T_{kj} \neq 0$, it is the case that $B_{jp} = 0$ for all $p \neq k$, and we conclude that T^{-1} has at most one nonzero entry in the jth row. Similarly, if $B_{jp} \neq 0$ for some p, then $T_{kj} = 0$ for all $k \neq p$ and T has at most one nonzero entry in the jth column. Both T and its inverse are isometries, and so there cannot be a column or row of all zeroes. Hence, T has exactly one nonzero entry per row and column.

For each j, let $\pi(j)$ be the unique positive integer for which $T_{j\pi(j)} \neq 0$. If j is given and $x = x_{\pi(j)} \in X_{\pi(j)}$, then $Tx = T_{j\pi(j)}x_{\pi(j)} \in X_j$, and (i) follows readily. Furthermore, if $x = \sum x_j \in X$, then $Tx = \sum (Tx)_j$, where $(Tx)_j = \sum_{k \geq 1} T_{jk}x_k = T_{j\pi(j)}x_{\pi(j}$. Since T is an isometry, for $x = x_{\pi(j)}$ we obtain

$$\|T_{j\pi(j)}x_{\pi(j)}\|_2 = \|Tx_{\pi(j)}\|_2 = \|x_{\pi(j)}\|_2.$$

This shows $T_{j\pi(j)}$ to be a unitary operator as advertised. Finally, it is clear that $\pi \in S_\mu$ since

$$\mu((\|x_j\|_2)) = \mu((\|T_{j\pi(j)}x_{\pi(j)}\|_2)) = \mu((\|x_{\pi(j)}\|_2))$$

so that (ii) is established. □

Again, we point out that the above theorem will apply to X if it is any complex admissible sequence space, or is a Banach space with a normalized one-unconditional Schauder basis.

The reader may recall (see Theorem 5.2.6., for example) that H is a Hermitian operator if and only if $\exp(itH)$ is an isometry for each $t \in \mathbb{R}$. Let $\mathcal{U}$ denote the family of all isometries of the form $U = \exp(iH)$ where H is Hermitian. Observe that if T is any isometry and $U \in \mathcal{U}$, then $T^{-1}\exp(iH)T = \exp(iT^{-1}HT)$, so that $\mathcal{U}$ is a normal subgroup of the group $\mathcal{G} = \mathcal{G}(X)$ of all surjective isometries on X. The fact that $\mathcal{U}$ is a group is clear from the nature of the operator matrix representation of the Hermitians given in Theorem 9.2.9. Let us collect some of this information in the next theorem.

9.2.13. THEOREM. *Let X be an admissible sequence space, let $\mathcal{G} = \mathcal{G}(X)$ be the group of surjective isometries on X, and let $\mathcal{U}$ be as defined above.*

(i) *The class $\mathcal{U}$ is a normal subgroup of $\mathcal{G}$.*
(ii) *If $U = [u_{jk}]$ is the matrix representation of the operator U with respect to the coordinate vector basis, then $U \in \mathcal{U}$ if and only if U is unitary as a Hilbert space operator and $u_{jk} = 0$ for $k \not\sim j$.*
(iii) *If T is an isometry on X, there exists $U \in \mathcal{U}$ and an operator permutation matrix P such that $T = UP$.*

PROOF. The argument for (i) was given prior to the statement of the theorem. For part (ii), we note that the operator matrix representation is diagonal and each entry $\exp(iH_{jj})$ is unitary. The rest follows from Corollary 9.2.11. Finally, given the description of T from Theorem 9.2.12 (ii), the isometry $T_{j\pi(j)}$ can be written as $\exp(iH_{jj})$ for some Hermitian H_{jj}. If P is the operator matrix for which $P_{j\pi(j)} = I$ for each j and for which the other entries are the zero matrix, and $U = \exp(iH)$ for the diagonal operator matrix $H = (H_{jj})$, the product UP is equal to T.

□

We note that in the case where each Hilbert subspace is of dimension 1, an isometry as given in (iii) above is the product of a diagonal matrix with modulus 1 scalars on the diagonal and a permutation matrix. We will refer to such an isometry as a *permutation isometry.*

9.3. Hermitian Elements and Orthonormal Systems

In this section we will develop the decomposition of a Banach space into what are called Hilbert components. There is great overlap with the previous section, but here the methods are more general and more elegant.

9.3.1. DEFINITION. *An element x of a complex Banach space X is said to be* Hermitian *if there is a Hermitian projection P_x onto the linear span of x. The set of Hermitian elements will be denoted by $h(X)$ and its closed linear span by $\hat{h}(X)$.*

If $x \in h(X)$ and φ is a duality map on X, we may suppose that it satisfies $\|x\|^2 P_x = \varphi(x) \otimes x = x^* \otimes x$. (We will find it often convenient to use the correspondence $x \to x^*$ to represent a duality map. As usual, $z \otimes x$ represents the rank 1 operator defined by $(z \otimes x)(y) = \langle\langle y, z\rangle\rangle x$.) For the s.i.p. $[\cdot,\cdot]$ determined by the duality map, this condition can be expressed also by requiring that for $x \in h(X)$, we have

$$[y, x][x, y] \in \mathbb{R} \tag{73}$$

for all $y \in X$. Note that with this assumption, $[x, y]$ will be unique for $y \in h(X)$.

We now observe some facts about Hermitian elements.

9.3.2. PROPOSITION. *For any Banach space (X, ν),*

(i) *the set $h(X)$ is closed, and*
(ii) *if $x, y \in h(X)$ and $[x, y] = 0$, then $[y, x] = 0$.*

PROOF. (i) Suppose x is the limit of a sequence $\{x_n\}$ in $h(X)$. Since $0 \in h(X)$, we may assume that $x \neq 0$. We may suppose that $\inf \nu(x_n) = t > 0$. Let $q_n = \nu(x_n)^{-2} x_n^*$. Then $q_n \otimes x_n$ is a Hermitian projection and $\nu^*(q_n) \le t^{-1}$. The functionals $\{q_n\}$ are bounded in X^* and by weak*-compactness, must have a w*-limit point q in X^*. For each $n \in \mathbb{N}$ we have

$$|1 - q_n(x)| = |q_n(x - x_n)| \le t^{-1}\nu(x - x_n).$$

Since the right-hand term goes to zero with n, we conclude that $q(x) = 1$ and $q \otimes x$ is a nonzero projection.

To show that $q \otimes x$ is Hermitian, it suffices to show that $exp(i\theta(q \otimes x))$ is an isometry. Given $z \in X$ and $\theta \in \mathbb{R}$, and using the fact that $q_n \otimes x_n$ is Hermitian, we have

$$\nu(z) = \nu(z + (e^{i\theta} - 1)q_n(z)x_n).$$

If we let $n \to \infty$, we obtain

$$\nu(z + (e^{i\theta} - 1)q(z)x) = \nu(z),$$

which shows that $q \otimes x$ is Hermitian and $x \in h(X)$.

(ii) If we suppose that $[x, y] = 0$, then

$$\nu(y)^2 P_y x = [x, y]x = 0,$$

and we must have $P_y x = 0$. Thus $P_y P_x = 0$, from which it follows that $P_x P_y = 0$ and $[y, x] = 0$. This last statement comes from known results about Hermitian operators [**42**, Theorem 2.13]. □

9.3.3. DEFINITION. *A collection $\{e_\alpha : \alpha \in \mathcal{A}\}$ is an* orthonormal system *if $[e_\alpha, e_\beta] = \delta_{\alpha\beta}$ for $\alpha, \beta \in \mathcal{A}$ and $\{e_\alpha : \alpha \in \mathcal{A}\} \subset h(X)$. An orthonormal system is said to be* complete *if its closed linear span is all of X.*

9.3.4. DEFINITION. *A closed subspace Y of a Banach space X is* orthonormal *if it is the closed linear span of an orthonormal system. If in addition Y is the range of a Hermitian projection, Y is said to be* orthogonal.

A collection $\{X_\alpha : \alpha \in \mathcal{A}\}$ *of orthogonal subspaces is* mutually orthogonal *if the associated Hermitian projections* $P_\alpha : X \to X_a$ *satisfy* $P_\alpha P_\beta = 0$ *if* $\alpha \neq \beta$. *(Such a collection of projections is sometimes called a* splitting *of* X. *The closed linear span of the union of the* $P_\alpha(X)$ *is called the* extent *of the splitting. If the extent of the splitting* $\{P_\alpha\}$ *is all of* X, *then* $\{P_a\}$ *is called a* Hermitian decomposition *of* X.*)*

At this point we need to make some observations about Hermitian splittings. First we note that a Hermitian projection P is necessarily of norm 1, since

$$1 = \| \exp(i\pi P)\| = \|I - 2P\| \geq 2\|P\| - 1.$$

Let $\{P_\alpha : \alpha \in \mathcal{A}\}$ be a splitting of X with extent Y. For any finite subset $\mathcal{F}$ of $\mathcal{A}$, $P_\mathcal{F} = \sum_{\alpha\in\mathcal{F}} P_\alpha$ is a Hermitian projection and so of norm 1. Hence, the set $\{P_\mathcal{F} : \mathcal{F} \subset \mathcal{A}\}$ is equicontinuous, from which it follows that $\{x \in X : P_\mathcal{F} x \to x\}$ is a closed linear subspace of X containing $P_\alpha(X)$ for each $\alpha \in \mathcal{A}$. Therefore, we see that for any $x \in Y$, $x = \sum\limits_{\alpha\in\mathcal{A}} P_\alpha x$. This means that a Hermitian decomposition is an unconditional Schauder decomposition.

9.3.5. Proposition. *(Kalton and Wood) Let* $\{P_\alpha : \alpha \in \mathcal{A}\}$ *be a splitting of* X, *and let* Y *be its extent. The following are equivalent.*

(i) *For* $x \in X$, $\sum_{\alpha\in\mathcal{A}} P_\alpha x$ *converges.*
(ii) *The space* Y *is the range of a Hermitian projection.*
(iii) *There is a Hermitian decomposition of* X *containing* $\{P_\alpha : \alpha \in \mathcal{A}\}$.
These conditions are implied by:
(iv) *The space* Y *contains no subspace isomorphic to* c_0.

Proof. (i)⇒ (ii). Define $Px = \sum\limits_{\alpha\in\mathcal{A}} P_\alpha x$.

(ii)⇒ (iii). If P is a Hermitian projection onto Y, add $I - P$ to the original collection of projections.

(iii)⇒ (i). If $\{P_\alpha : \alpha \in \mathcal{A}\} \cup \{P_\beta : \beta \in \mathcal{B}\}$ is a Hermitian decomposition of X, then for each $x \in X$, unordered convergence of $\sum\limits_{\mathcal{A}\cup\mathcal{B}} P_\alpha x$ implies the same for $\sum\limits_{\alpha\in\mathcal{A}} P_\alpha x$.

(iv) ⇒ (i). Suppose $\sum P_\alpha x$ does not converge. In that case there is a subseries $\sum\limits_{j=1}^{\infty} P_{\alpha_j} x$ which does not converge. However, this series is weakly unconditionally Cauchy since the partial sums are uniformly bounded. (For any bounded real sequence $\{t_n\}$ going to zero show $\sum t_n(P_{\alpha_j} x)$ converges and apply Lemma 2 of [**46**].) By Theorem 5 of [**46**], Y contains a subspace isomorphic to c_0. □

9.3.6. PROPOSITION. *(Kalton and Wood) If P and Q are Hermitian projections on X such that $PQ = 0$, then both $PTQ + QTP$ and $i(PTQ - QTP)$ are Hermitian whenever T is Hermitian.*

PROOF. First we argue by induction that if $P_1, P_2, \ldots, P_n$ form a Hermitian decomposition of X, then for any Hermitian T, we must have that $\sum P_j T P_j$ is also Hermitian. For this let

$$S_p = \sum_{j=1}^{n-p} \sum_{k=1}^{n-p} P_j T P_k + \sum_{j=n-p+1}^{n} P_j T P_j \quad (0 \le p \le n).$$

Note that $S_0 = T$, and assume that S_q is Hermitian. Now $I - 2P_{n-q}$ is an isometry and is also its own inverse. Since

$$exp(it(I - 2P_{n-q})S_q(I - 2P_{n-q})) = (I - 2P_{n-q}exp(itS_q))(I - 2P_{n-q}),$$

we have that $(I - 2P_{n-q})S_q(I - 2P_{n-q})$ is Hermitian. Hence, we may conclude that

$$S_{q+1} = \frac{1}{2}[S_q + (I - 2P_{n-q})S_q(I - 2P_{n-q})]$$

is Hermitian and so by induction, S_n is Hermitian as claimed.

Now given P, Q as in the statement of the proposition, and letting $R = I - (P + Q)$, we can apply the result just proved above to the decompositions $(P + Q, R)$ and (P, Q, R) to obtain the fact that

$$PTP + QTQ + RTR \quad \text{and} \quad PTP + QTQ + RTR + PTQ + QTP$$

are Hermitian. Therefore, $PTQ + QTP$ is Hermitian. By a known result about Hermitian operators [**49**, Lemma 4, p. 47],

$$i[P(PTQ + QTP) - (PTQ + QTP)P] = i(PTQ - QTP)$$

is Hermitian. □

It is clear that an orthonormal system $\{e_\alpha : \alpha \in \mathcal{A}\}$ naturally gives rise to a Hermitian splitting $\{P_\alpha\}$ defined by $P_\alpha x = [x, e_\alpha]e_\alpha$. The next proposition is important to the theory of orthonormal systems, but it is essentially a restatement of Proposition 9.3.5.

9.3.7. PROPOSITION. *(Kalton and Wood) Let $\{e_\alpha : \alpha \in \mathcal{A}\}$ be an orthonormal system which has extent Y. Then*

(i) $\sum_{\alpha \in \mathcal{A}} [x, e_\alpha]e_\alpha$ *converges for $x \in Y$,*
(ii) $\sum_{\alpha \in \mathcal{A}} [x, e_\alpha]e_\alpha$ *converges for all $x \in X$ if and only if Y is orthogonal,*
(iii) *if Y contains no subspace isomorphic to c_0, then Y is orthogonal.*

Observe that an orthonormal basis $\{e_\alpha : \alpha \in \mathcal{A}\}$ is actually hyperorthogonal. This can be shown directly by first showing that the norm is unchanged if the coordinate $[x, e_\alpha]$ is replaced by its absolute value. This follows since

$I - (e^{it} - 1)(e_\alpha^* \otimes e_\alpha)$ is an isometry for all real t. Then induction plus a limiting process give $\nu(x) = \nu(|x|)$, where by $|x|$, we mean $\sum |[x, e_\alpha]| e_\alpha$.

Our goal now is to show how an orthonormal basis induces a decomposition of the space into what will be called Hilbert components. This decomposition is essentially the same as was discussed in Section 9.2. The next result is crucial to the description we wish to obtain.

9.3.8. PROPOSITION. *(Kalton and Wood) Let (X, ν) be a Banach space and $\{e_1, e_2\}$ an orthonormal system in X. Suppose that there is a Hermitian operator T such that $[Te_1, e_2] \neq 0$. If $A = [a_{ij}]$ is a two-by-two Hermitian matrix (in the usual sense), then the operator $\sum_{j,k} a_{jk} e_j^* \otimes e_k$ is Hermitian, and for $a_1, a_2 \in \mathbb{C}$,*

$$\nu(a_1 e_1 + a_2 e_2)^2 = |a_1|^2 + |a_2|^2.$$

Furthermore, the linear hull, $sp\{e_1, e_2\}$, is contained in $h(X)$.

The same conclusions hold if the statement about the operator T is replaced by the assumption that $ae_1 + be_2 \in h(X)$ for some $a, b \in \mathbb{C}$ with $ab \neq 0$.

PROOF. Let $a = [Te_1, e_2]$ and $b = [Te_2, e_1]$. Then

$$ae_1^* \otimes e_2 + be_2^* \otimes e_1 = (e_2^* \otimes e_2)T(e_1^* \otimes e_1) + (e_1^* \otimes e_1)T(e_2^* \otimes e_2)$$

is Hermitian by Proposition 9.3.6. Restricting that operator to $Y = sp\{e_1, e_2\}$, its matrix representation will be given by $\left[\begin{smallmatrix} 0 & b \\ a & 0 \end{smallmatrix}\right]$, and so (by Theorem 9.2.11), $b = \bar{a}$. Similarly, by 9.3.6 again, $i(ae_1^* \otimes e_2 - \bar{a}e_2^* \otimes e_1)$ is Hermitian, from which it follows that both $e_1^* \otimes e_2 + e_2^* \otimes e_1$ and $i(e_1^* \otimes e_2 - e_2^* \otimes e_1)$ are Hermitian. If $A = [a_{jk}]$ is a Hermitian matrix, then $a_{21} = \overline{a_{12}}$ and $a_{12}(e_1^* \otimes e_2) + \overline{a_{12}}(e_2^* \otimes e_1)$ is easily shown to be a real linear combination of $e_1^* \otimes e_2 + e_2^* \otimes e_1$ and $i(e_1^* \otimes e_2 - e_2^* \otimes e_1)$. It should now be clear that the first part of the assertion is true.

Suppose $a_1, a_2 \in \mathbb{C}$ with $|a_1|^2 + |a_2|^2 = 1$. Then the matrix

$$U = \begin{bmatrix} a_1 & -\overline{a_2} \\ a_2 & \overline{a_1} \end{bmatrix}$$

is unitary, and so $U = exp(iA)$, where $A = [a_{jk}]$ is ordinary Hermitian. Thus $W = \sum a_{jk}(e_j^* \otimes e_k)$ is a Hermitian operator and $exp(iW)$ is an isometry on X. The matrix representing W restricted to $Y = sp\{e_1, e_2\}$ is the same as A, so that the matrix U above represents an isometry on Y. But U represents the operator

$$a_1(e_1^* \otimes e_1) + a_2(e_1^* \otimes e_2) - \overline{a_2}(e_2^* \otimes e_1) + \overline{a_1}(e_2^* \otimes e_2)$$

which is, therefore, an isometry on Y. If we apply this operator to e_1, we get $\nu(a_1 e_1 + a_2 e_2) = 1$, from which the general statement follows.

Finally, suppose $x_0 = a_1e_1 + a_2e_2 \in Y$ with $|a_1|^2 + |a_2|^2 = 1$. The matrix $A = [a_j\overline{a_k}]$ is ordinary Hermitian, so that by the first part of the proof,

$$W = |a_1|^2((e_1^* \otimes e_1) + a_1\overline{a_2}(e_1^* \otimes e_2) + \overline{a_1}a_2(e_2^* \otimes e_1) + |a_2|^2(e_2^* \otimes e_2)$$

is Hermitian. Since W is also a projection onto the span of x_0, we have that $x_0 \in h(X)$. It is clear now that $Y \subset h(X)$.

To justify the very last statement in the proposition, suppose $ae_1 + be_2 \in h(X)$ with $ab \neq 0$, let P be the Hermitian projection onto $sp\{ae_1 + be_2\}$ and let P_0 be its restriction to Y. Now the matrix representing P_0 relative to e_1, e_2 cannot be diagonal, from which we conclude that either $[Pe_1, e_2] \neq 0$ or $[Pe_2, e_1] \neq 0$. The desired conclusions then follow from the previous argument. □

9.3.9. Corollary. *(Kalton and Wood) If $h(X) = X$, then X is a Hilbert space.*

Proof. Since it suffices to show that the norm satisfies the parallelogram law, consider any two-dimensional subspace and select a basis which satisfies the hypotheses of Proposition 9.3.8. □

We are now ready to talk about Hilbert components. Obviously, if $x \in h(X)$, then $sp\{x\} \subset h(X)$, so every such x is contained in a maximal subspace of $h(X)$.

9.3.10. Definition. *Let $\{H_\lambda : \lambda \in \Lambda\}$ be the collection of maximal linear subspaces of $h(X)$. These are called the Hilbert components of X.*

Since $h(X)$ is closed, each H_λ is closed, and $h(H_\lambda) = H_\lambda$. By the corollary, H_λ is a Hilbert space.

9.3.11. Lemma. *(Kalton and Wood) If $x, y \in h(X)$ and $[x, y] \neq 0$, then $sp\{x, y\} \subset h(X)$.*

Proof. Since if x, y are linearly dependent, the conclusion is obvious, we may suppose they are independent. Let $e_1 = \nu(x)^{-1}x$ and $f = y - [y, e_1]e_1$, which must be nonzero. Let $P = e_1^* \otimes e_1$ and $Q = I - P$, so that by the first part of the proof of Proposition 9.3.6, $S = P(y^* \otimes y)P + Q(y^* \otimes y)Q$ is Hermitian. Furthermore, we have that $P(y^* \otimes y)P = \lambda P$, where $\lambda = [e_1, y][y, e_1]$ is real. Thus, S and $P(y^* \otimes y)P$ are both Hermitian, from which we conclude that their difference $Q(y^* \otimes y)Q = Q^*y^* \otimes Qy$ is also Hermitian.

We obtain from Proposition 9.3.6 that $Q(y^* \otimes y)P + P(y^* \otimes y)Q$ and $i[Q(y^* \otimes y)P - P(y^* \otimes y)Q]$ are both Hermitian. If $Q^*y^* = 0$, then $P(y^* \otimes y)Q = 0$ and it follows that $Q(y^* \otimes y)P = P^*y^* \otimes Qy$ and $i(P^*y^* \otimes Qy)$ are both Hermitian. This implies that $P^*y^* \otimes Qy = 0$, from which we would have $P^*y^* = 0$. This is a contradiction to the hypothesis that $[x, y] = P^*y^*x \neq 0$. Hence, we must have $Q^*y^* \neq 0$ so that $f = Qy \in h(X)$ by means of the nonzero Hermitian projection $Q^*y^* \otimes Qy$. If we let $e_2 = \nu(f)^{-1}f$, then e_1, e_2 satisfy the hypotheses of Proposition 9.3.8, so that $sp\{x, y\} = sp\{e_1, e_2\} \subset h(X)$. □

9.3.12. THEOREM. *(Kalton and Wood) If (X, ν) is a complex Banach space, the collection $\{H_\lambda : \lambda \in \Lambda\}$ of Hilbert components forms a mutually orthogonal collection of subspaces of X. The union of the collection of orthonormal bases, one from each component, forms a maximal orthonormal system whose closed linear span is $\hat{h}(X)$. The system is complete if and only if $\hat{h}(X) = X$.*

PROOF. Each H_λ is the extent of an orthonormal system since it is a Hilbert space and possesses an orthonormal basis. Furthermore it must be orthogonal by Proposition 9.3.7, since it has no subspace isomorphic to c_0. Suppose $x \in H_\lambda, y \in H_\tau$, with $\lambda \neq \tau$, and $[y, x] \neq 0$. By Proposition 9.3.11, $sp\{x, y\} \subset h(X)$. Let $z \in H_\tau$ be different from y. If $[x, z] \neq 0$, then $sp\{x, z\} \subset h(X)$. Otherwise, we may choose $r > 0$ small enough that $[y + rz, x] \neq 0$ and $[y + rz, z] \neq 0$. Let $P = \nu(y + rz)^{-2}(y + rz)^* \otimes (y + rz)$ be the Hermitian projection associated with the Hermitian element $y + rz$. Then $[Px, z] \neq 0$, and it follows from Proposition 9.3.8 that $sp\{x, z\} \subset H_\tau$. We now see from the maximality of H_τ that $x \in H_\tau$, so that $H_\lambda \subset H_\tau$. Clearly, a similar argument would show the opposite containment so that $H_\lambda = H_\tau$. This contradiction shows that we must have $[x, y] = 0$, which proves the statement about mutual orthogonality.

The final two statements are easily seen to be true. □

If $\{e_\alpha : \alpha \in \mathcal{A}\}$ is an orthonormal basis for X, the corresponding family of Hilbert components $\{H_\lambda : \lambda \in \Lambda\}$ with the associated Hermitian projections $\{P_\lambda\}$ provide an Hermitian decomposition of X. In the spirit of Section 9.2, we can say that for $j, k \in \mathcal{A}$, j is equivalent to k (written $j \sim k$) if $\nu(x) = \nu(y)$ whenever $|[x, e_j]|^2 + |[x, e_k]|^2 = |[y, e_j]|^2 + |[y, e_k]|^2$ and $|[x, e_\alpha]| = |[y, e_\alpha]|$ for all $\alpha \in \mathcal{A}$ with $\alpha \neq j, k$.

We will show now that the decomposition into Hilbert components that we have seen above is essentially the same as that obtained for sequence spaces in Section 9.2.

9.3.13. THEOREM. *Let $\{e_\alpha : \alpha \in \mathcal{A}\}$ be an orthonormal basis for X and suppose that $\{H_\lambda : \lambda \in \Lambda\}$ are the associated Hilbert components.*

(i) *If $j, k \in \mathcal{A}$, then $j \sim k$ if and only if $e_j, e_k \in H_\lambda$ for some $\lambda \in \Lambda$.*

(ii) *If $x, y \in X$ so that $\|x_\lambda\|_2 = \|y_\lambda\|_2$ for each $\lambda \in \Lambda$, then $\nu(x) = \nu(y)$.*

PROOF. (i) First suppose that $e_j, e_k \in H_\lambda$. By Proposition 9.3.8, the operator $A = i[(e_j^* \otimes e_k) - (e_k^* \otimes e_j)]$ is Hermitian, and $exp(itA) = I + i \sin tA + (\cos t - 1)P$ is an isometry, where $P = (e_1^* \otimes e_1) + (e_2^* \otimes e_2)$ is a Hermitian projection. Now

$$exp(itA)(|x|) = (|[x, e_k]| \sin t + |[x, e_j]|)e_j + (|[x, e_k] \cos t - |[x, e_j]|)e_k + \sum_{\alpha \neq j,k} |[x, e_\alpha]|e_\alpha \tag{74}$$

has the same norm as x. If $|[y, e_j]|^2 + |[y, e_k]|^2 = |[x, e_j]|^2 + |[x, e_k]|^2$, and $|[y, e_\alpha]| = |[x, e_\alpha|$ for all $\alpha \neq j, k$, then there exists $t \in \mathbb{R}$ such that $|[y, e_j]|$ and

$|[y, e_k]|$ are equal to the coefficients of e_j and e_k, respectively, in the right-hand side of (74). Hence $\nu(y) = \nu(x)$, and we conclude that $j \sim k$.

Next suppose that $j \sim k$ and $e_k \in H_\lambda$. Let $z \in H_\lambda$ with $\nu(z) = 1$. If $[z, e_j] \neq 0$, then $sp\{x, e_j)\} \subset h(X)$ by Lemma 9.3.11. Suppose $[z, e_j] = 0$. Now there must be some $m \in \mathcal{A}$ so that $[z, e_m] \neq 0$ where $e_m \in H_\lambda$. Then by the first part of this proof and the transitive property of the equivalence relation, we have $j \sim m$. A straightforward calculation shows that $exp(itA)$ is an isometry for every real t, where $A = e_j^* \otimes e_m + e_m^* \otimes e_j$. Thus A is Hermitian and $[Az, e_m] = [z, e_j] \neq 0$. By Proposition 9.3.8, $sp\{e_j, z\} \subset h(X)$. The maximality of H_λ requires that $e_j \in H_\lambda$.

(ii) Consider an element $x \in X$ with Schauder decomposition expansion given by $x = \sum_{\lambda \in \Lambda} x_\lambda$, where $x_\lambda = P_\lambda x$, as given just above the statement of Proposition 9.3.5. Let $\tau \in \Lambda$ be fixed and suppose $x_\tau = \sum_{k \in \Gamma(\tau)} [x, e_k]e_k$, where $\Gamma(\tau)$ is the set of all $k \in \mathcal{A}$ for which $e_k \in H_\tau$. Let $j \in \Gamma(\tau)$ be fixed also. Then by the same argument as given in the beginning of this proof (or as in the proof of Theorem 9.2.4), we see that

$$\nu(x) = \nu(\sum_{\lambda \neq \tau} x_\lambda + (|[x, e_j]|^2 + |[x, e_m]|^2)e_j + \sum_{k \neq j, m} [x, e_k]e_k).$$

This can be extended to any finite sum (in the second term above), and by a limiting process to obtain

$$\nu(x) = \nu(\sum_{\lambda \neq \tau} x_\lambda + \|x_\tau\|_2 e_j).$$

It follows that if $x, y \in X$ with $y_\lambda = x_\lambda$ for $\lambda \neq \tau$, and $\|y_\tau\|_2 = \|x_\tau\|_2$, then $\nu(x) = \nu(y)$. This can be extended by induction and then a limiting process to obtain the conclusion in (ii).

□

Let us now obtain the characterizations of Hermitian operators and isometries in this setting. This time, we will characterize the isometries first.

9.3.14. THEOREM. *(Kalton and Wood) Suppose X is a complex Banach space with $h(X) \neq \{0\}$. Let $\{H_\lambda : \lambda \in \Lambda\}$ be the Hilbert components of X and $\{P_\lambda : \lambda \in \Lambda\}$ the associated Hermitian projections.*

(i) *If $U : X \to X$ is an isometry, then $U(h(X)) = h(X)$ and there is a bijection $\gamma : \Lambda \to \Lambda$ such that $U(H_\lambda) = H_{\gamma(\lambda)}$.*
(ii) *If T is Hermitian, then $TP_\lambda = P_\lambda T$ for all $\lambda \in \Lambda$.*

PROOF.
(i) If $x \in h(X)$, with $\nu(x) = 1$, then $U(x^* \otimes x)U^{-1} = (U^{-1})^* x^* \otimes Ux$ is a Hermitian projection which shows that $Ux \in h(X)$. Thus $U(h(X)) \subset h(X)$, and using U^{-1}, we can get the opposite containment so that $U(h(X)) = h(X)$.

For a given $\lambda \in \Lambda$, $U(H_\lambda)$ is a subspace of $h(X)$, and so must be contained in H_τ for some τ. Since $U^{-1}(H_\tau) \subset H_\kappa$ for some κ, we get $H_\lambda \subset U^{-1}(H_\tau) \subset H_\kappa$ and must conclude that $\lambda = \kappa$. Thus $U(H_\lambda) = H_\tau$, and we define γ by $\gamma(\lambda) = \tau$.

(ii) If $t \in \mathbb{R}$ and T is Hermitian, then $exp(itT)$ is an isometry, and we have from part (i) that $exp(itT)(H_\lambda) = H_{\kappa(t)}$. For $x \in H_\lambda$ and t such that $\kappa(t) \neq \lambda$,

$$\nu([exp(itT) - I]x) \geq \nu(P_\lambda[exp(itT) - I]x) = \nu(x).$$

Since $\lim_{t\to 0} \|exp(itT) - I)\| = 0$, we must have $\kappa(t) = \lambda$ for small t, for otherwise, $\|exp(itT) - I\| \geq 1$. Then

$$exp(-itT)P_\lambda exp(itT) = P_\lambda$$

and upon expansion, we see that $TP_\lambda = P_\lambda T$. □

Next we characterize the Hermitian operators on X in the case where X has an orthonormal basis.

9.3.15. THEOREM. *(Kalton and Wood) Suppose X is a complex Banach space with an orthonormal basis. Let $\{H_\lambda : \lambda \in \Lambda\}$ be the Hilbert components of X. A bounded operator T is Hermitian if and only if $T(H_\lambda) \subset H_\lambda$ and T is Hermitian as an operator on the Hilbert space H_λ, for each $\lambda \in \Lambda$.*

PROOF. If T is Hermitian, it follows from part (ii) of Theorem 9.3.14 that $T(H_\lambda) \subset H_\lambda$. Furthermore, the isometry $exp(itT)$ restricted to H_λ shows that T restricted to H_λ must be Hermitian as an operator on that Hilbert space.

For the converse, suppose that $T(H_\lambda) \subset H_\lambda$ and T is Hermitian on H_λ. By spectral theory, for each λ, there is a net $\{S_{n,\lambda}\}$ of finite rank Hermitian operators on H_λ which converges to T in the weak operator topology of H_λ. Furthermore, we may choose on orthonormal system $\{e_j : 1 \leq j \leq k(n)\}$ and real numbers a_j such that

$$S_{n,j} = \sum_{j=1}^{k(n)} a_j e_j^* \otimes e_j.$$

Now $S_{n,\lambda}P_\lambda : X \to X$ is Hermitian, and by taking weak operator limits, we get that TP_λ is also Hermitian. Again by taking weak operator limits, $T = \sum TP_\lambda$ is Hermitian. □

9.4. The Case for Real Scalars: Functional Hilbertian Sums

In the previous sections our methods required that the Banach spaces under consideration were over the complex scalars. In the present section we will examine Rosenthal's approach to obtain orthogonal decompositions for real Banach spaces. The key player in the earlier discussions was the Hermitian operator, whose very definition would appear meaningless for a real space. That role will be played in this section by what is called a *skew-Hermitian* operator. An operator T on a Banach space (X, μ) is said to be *skew-Hermitian* if $\Re x^*(Tx) = 0$ for all $x \in X$ and x^* dual to x in X^*. Note that this definition is equivalent to saying that $\Re[Tx, x] = 0$ for any s.i.p. compatible with the norm μ on X, and that it makes sense whether X is real or complex. We will denote the set of all skew-Hermitian operators on X by $\mathfrak{A}(X)$, which we refer to as the Lie algebra of X. Observe that if X is a finite dimensional real Euclidean space, then the ordinary skew-symmetric matrices (i.e., $M^t = -M$) are examples of skew-Hermitian operators. Also, it is clear that in the complex case, an operator T is Hermitian if and only if $iT \in \mathfrak{A}(X)$. The use of the term *Lie algebra* is suggestive of the fact that it is closed under the Lie product $TS - ST$. Let us investigate more formally some important properties of skew-Hermitian operators. As earlier, we will let $\mathcal{G}(X)$ denote the group of surjective isometries on X.

Our first result is an analogue of Theorem 5.2.6 from Chapter 5 applied to real spaces. The proof is essentially the same as that of Theorem 5.2.6, since Lemma 5.2.5 of Chapter 5 holds in this setting. Let us also note that everything in this section is due to Rosenthal.

9.4.1. PROPOSITION. *Suppose that T is a bounded linear operator on X, where X is a real Banach space. The following are equivalent:*

(i) $T \in \mathfrak{A}(X)$.
(ii) $\|\exp(tT)\| \leq 1$ *for all real* t.
(iii) $\exp(tT) \in \mathcal{G}(X)$ *for all real* t.
(iv) $\displaystyle\lim_{t\to 0} \frac{\|I + tT\|}{t} = 0$.

9.4.2. PROPOSITION. *Let S and T be skew-Hermitian operators on X and $U \in \mathcal{G}(X)$. Suppose Z is a subspace of X and P is a norm 1 projection from X onto Z.*

(i) $U^{-1}TU$ *is skew-Hermitian.*
(ii) $\mathfrak{A}(X)$ *is a weak-operator-closed real linear space.*
(iii) $TS - ST$ *is skew-Hermitian.*
(iv) $PT|Z$ *is in* $\mathfrak{A}(Z)$.

PROOF. We will assume that X is a real Banach space. Otherwise, if X is complex and $X_{\mathbb{R}}$ is X considered as a real space, it is clear that $T \in \mathfrak{A}(X)$ if and only if it is also in $\mathfrak{A}(X_{\mathbb{R}})$.

(i) If x^* is dual to x, it is straightforward to show that $(U^{-1})^*x^*$ is dual to Ux. Hence

$$[U^{-1}TUx, x] = \langle\langle U^{-1}TUx, x^*\rangle\rangle = \langle\langle TUx, (U^{-1})^*x^*\rangle\rangle = [TUx, Ux] = 0$$

for every x, where $[\cdot, \cdot]$ is a compatible s.i.p.

(ii) This follows directly from real linearity of the left-hand argument of the s.i.p. and from the definition of the weak-operator topology.

(iii) Define f from $\mathbb{R}$ into the bounded linear operators on X by

$$f(t) = e^{-tS}Te^{tS}$$

for all real t. Then $f(t) \in \mathfrak{A}(X)$ from (i) since e^{tS} is an isometry by 9.4.1 (iii). Now $f'(t) \in \mathfrak{A}(X)$ by (ii) above. Since

$$f'(t) = e^{-tS}TSe^{tS} - Se^{-tS}Te^{tS},$$

it follows that $f'(0) = TS - ST$ is skew-Hermitian.

(iv) This is true as a result of the fact that if z^* is dual to z in Z, then P^*z^* is dual to z as an element of X.

□

In the previous section we defined a closed subspace X of a Banach space Z to be *orthogonal* if it was the closed linear span of an orthonormal system and the range of a Hermitian projection P. If Y is the complement of X, it follows from the fact that $\exp(itP)$ is an isometry that $\|x+y\| = \|rx+y\| = \|x+sy\| = \|rx+sy\|$, where r, s are scalars with $|r| = |s| = 1$. We use this idea to extend the definition to the case where the scalars are real. We also take this opportunity to define some other terms that will be useful as we proceed.

9.4.3. Definition. *Let Z denote a Banach space with X, Y subspaces of Z.*

(i) *We say that X is* orthogonally complemented *and that Y is an* orthogonal complement *of X if $Z = X + Y$ and for all $x \in X, y \in Y$ and scalars r, s with $|r| = |s| = 1$ we have $\|x+y\| = \|rx+sy\|$. We denote Y by $\mathcal{O}(X)$. The projection P with range X and kernel Y is called the orthogonal projection on X.*

(ii) *The subspaces X and Y are said to be* orthogonal *if $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ both exist with $X \subset \mathcal{O}(Y)$ (equivalently $Y \subset \mathcal{O}(X)$).*

(iii) *The subspace X will be said to be* well embedded *in Z if $Z = X + Y$ and $\|x+y\| = \|Ux+y\|$ for all $x \in X$, $y \in Y$, and U a surjective isometry on X.*

(iv) *The subspace X is said to be a* Hilbert component *if X is a maximal nonzero well-embedded Hilbert subspace. We let $\mathcal{H} = \mathcal{H}(Z)$ denote the collection of Hilbert components of Z.*

We should note here that it can be shown that orthogonal complements are unique so that it is appropriate to use the language "the orthogonal complement." If Y is the orthogonal complement of X as in (i) above, then $U(x+y) = x - y$ defines an isometry on $Z = X \oplus Y$, and $P = (U+I)/2$ defines an orthogonal projection of Z onto X with kernel Y for which $\|P\| = 1$. Hence $\|x\| \leq \|x+ry\|$ for all $r \in \mathbb{R}$ and we see that orthogonality here corresponds with the usual notion of orthogonality in Banach spaces. Also, we note that in the complex case, the projection P would be Hermitian, but it is not true that P is skew-Hermitian. The notion of Hilbert component here is, as we shall see, the same as that discussed in the previous section.

9.4.4. Proposition. *Let X be a nonzero Hilbert subspace of Z (i.e., a subspace of Z which is isometric to a Hilbert space) and Y a subspace such* *that $Z = X \oplus Y$. If Z is real, assume also that dim $X > 1$. The following are equivalent.*

(i) *X is a well-embedded Hilbert space with $\mathcal{O}(X) = Y$.*
(ii) *For all x, x' in X and y in Z, if $\|x\| = \|x'\|$ then $\|x+y\| = \|x'+y\|$.*
(iii) *For all $T \in \mathfrak{A}(X)$, $T \oplus 0 \in \mathfrak{A}(Z)$.*

We remark that the notation $T \oplus S$ for T, S, bounded linear operators on X, Y, respectively, means that $(T \oplus S)(x \oplus y) = Tx + Sy$ for $x \in X$ and $y \in Y$.

Proof. (i)$\Rightarrow$ (iii). Let $T \in \mathfrak{A}(X)$ and $r \in \mathbb{R}$. From 9.4.1 we see that e^{rT} is an isometry (in the complex case, we can think of $T \in \mathfrak{A}(X_{\mathbb{R}})$). Since X is well embedded, we have $\|(e^{rT} \oplus I)(x \oplus y)\| = \|e^{rT}(x) + y\| = \|x + y\|$, and $e^{rT} \oplus I$ is an isometry on Z. Since $e^{rT} \oplus I = e^{r(T \oplus 0)}$, we have $T \oplus 0$ is in $\mathfrak{A}(Z)$ by 9.4.1 again.

(iii)$\Rightarrow$ (ii). Suppose $x, x' \in X$ with $\|x\| = \|x'\|$. If dim $X \geq 2$, we may choose a two-dimensional subspace W of X containing x, x' and a rank 2 $T \in \mathfrak{A}(X)$ such that $TX \subset W$ and $e^T x = x'$. Now by hypothesis, $T \oplus 0$ is in $\mathfrak{A}(Z)$ so that $e^{T \oplus 0}$ is an isometry on Z. Thus given $y \in Y$, $\|x + y\| = \|e^{T\oplus 0}(x+y)\| = \|x' + y\|$. If Z is complex and X is one-dimensional, define T on X by $Tx = ix$. Since $x' = (\cos\theta + i\sin\theta)x$ for some θ between 0 and 2π, $e^{\theta T}$ is an isometry on X which takes x to x'. Then $e^{\theta T \oplus 0}$ is an isometry on Z and the conclusion follows as above.

(ii) $\Rightarrow$ (i). Let U be a surjective isometry on X and let $x \in X$, $y \in Y$. If we let $x' = Ux$, we have by the hypothesis that $\|Ux + y\| = \|x + y\|$ and we conclude that X is well embedded.

□

It is clear that any one-dimensional subspace that is orthogonally complemented must be well embedded. In the complex case such a space is generated by what we earlier called an Hermitian element, a member of $h(X)$. It is true that any well-embedded Hilbert subspace of X is contained in a maximal such space; that is, what we defined above as a Hilbert component. We will not prove that result here, nor will we prove the interesting fact that if X is a subspace of Z and every one-dimensional subspace of X is orthogonally complemented in Z, then X must itself be a well-embedded Hilbert subspace of Z. When $X = Z$ this gives a characterization of Hilbert space. Also it corresponds to Corollary 9.3.9 in the complex case, with the well-embedded part inherent in the proof of Theorem 9.3.13.

We will see that rank 2 skew-Hermitian operators play a significant role in our current discussion. In fact, the range of such an operator is necessarily a well-embedded Hilbert space.

9.4.5. Proposition. *Let Z be a real Banach space and suppose $T \in \mathfrak{A}(Z)$ is rank 2. Then $X = TZ$ is a well-embedded Hilbert space with orthogonal complement $Y = \ker T$.*

PROOF. Let us define 2×2 matrices R_0 and $R(\theta)$ for all real θ by

$$R_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ and } R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Suppose now that T, X, Y are as in the statement of the proposition. Since X is two-dimensional, it is known that there is a compatible Euclidean norm $\|\cdot\|_E$ on X. This means that $H = (X, \|\cdot\|_E)$ is a Hilbert space and that $\mathcal{G}(X) \subset \mathcal{G}(H)$. Hence, $T|X \in \mathfrak{A}(H)$. Any skew-Hermitian matrix on H, however, must be of the form cR_0, where c may be taken as one. Since $e^{\theta T}$ is a surjective isometry on Z for all θ, $e^{\theta R_0} = R(\theta)$ describes the surjective isometries on both X and H, so that X must itself be a Hilbert space. Furthermore, $T|X = W$ is invertible on X. Hence, given $z \in Z$, let $x = W^{-1}(Tz) \in X$ so that $T(z - x) = 0$ and we see that $Z = X + Y$. We now apply Proposition 9.4.4 to conclude that X is well embedded. □

There is a useful description of the orthogonal complement of a well-embedded Hilbert subspace. Let us state it carefully and sketch the proof.

9.4.6. PROPOSITION. *Let Z be a real Banach space and suppose X is a well-embedded Hilbert subspace of dimension at least 2. Then*

$$\mathcal{O}(X) = \cap\{\ker T : TZ \subset X, \ \textit{rank } T = 2 \ \textit{and } T \in \mathfrak{A}(Z)\}.$$

PROOF. Let W denote the set on the right above and suppose T is as described in the definition of W. Then TZ is well embedded and $\mathcal{O}(X) \subset \mathcal{O}(TZ) = \ker T$. Hence, $\mathcal{O}(X) \subset W$.

On the other hand, let $w \in W$, and choose $x \in X, y \in \mathcal{O}(X)$ so that $w = x + y$. If $x \neq 0$, there is a rank 2 operator $T \in \mathfrak{A}(Z)$ with $x \in TZ \subset X$. (This is possible since X is well embedded.) Now we must have $Tx \neq 0$, and since $\mathcal{O}(X) \subset \mathcal{O}(TZ) = \ker T$, $Ty = 0$. It follows that $Tw = Tx \neq 0$, which contradicts the fact that $w \in W$. We must conclude that $x = 0$ and $w = y \in \mathcal{O}(X)$. □

An important fact is that distinct Hilbert components are orthogonal. To prove that we will need the following lemma. We omit the proof, not because it is easy, but rather because it is somewhat lengthy. See the notes at the end of the chapter for a reference.

9.4.7. LEMMA. *(Rosenthal) Let Z be a real Banach space, X a nonzero well-embedded Hilbert subspace of Z, and T a rank 2 member of $\mathfrak{A}(Z)$. Then either $TZ \subset X$, $TZ \subset \mathcal{O}(X)$, or there exists a well-embedded Hilbert subspace Y which properly contains X.*

9.4.8. THEOREM. *(Rosenthal) Let H_1 and H_2 be distinct Hilbert components of a real Banach space Z with dim $H_1 \geq 2$. Then H_1 and H_2 are orthogonal.*

PROOF. It will suffice to show that $H_2 \subset \mathcal{O}(H_1)$. By Proposition 9.4.6 we know that

$$\mathcal{O}(H_1) = \cap\{\ker T : TZ \subset H_1, \ \text{rank} T = 2 \text{ and } T \in \mathfrak{A}(Z)\}.$$

If $H_2 \not\subseteq \mathcal{O}(H_1)$, we can choose $T \in \mathfrak{A}(Z)$ with rank 2 such that $TZ \subset H_1$ yet $H_2 \not\subseteq \ker T$. This shows that TZ is not contained in $\mathcal{O}(H_2)$, and since H_2 is a Hilbert component, it follows by Lemma 9.4.7 that $TZ \subset H_2$. Thus it is the case that $H_1 \cap H_2 \neq \emptyset$, and, in fact, that it is of dimension at least 2.

Choose a nonzero $x \in H_1 \cap H_2$, $y \in H_2 \backslash H_1$ and $S \in \mathfrak{A}(Z)$ with range equal to $sp\{x, y\}$. This is possible since $sp\{x, y\}$ is well embedded in H_2 and therefore in Z. Now $SZ \not\subseteq H_1$, but S is invertible on $sp\{x, y\}$. Hence $Sx \neq 0$ and $H_1 \not\subseteq \ker S$ so we must conclude that $SZ \not\subseteq \mathcal{O}(H_1)$. Once again we apply Lemma 9.4.7 to obtain a contradiction to the fact that H_1 is a Hilbert component. □

We now wish to show how real Banach spaces can be given a direct sum structure analogous to what we have observed in previous sections, and to characterize the skew-Hermitian operators and isometries on these spaces.

We first introduce another form of direct sum notation useful in this context, although it is really just a different way to describe a special class of substitution spaces $E(X_\alpha)_{\alpha\in\mathcal{A}}$. To begin with, in analogy with the notion of Hermitian decomposition as described in Section 3, a family $(X_\alpha)_{\alpha\in\mathcal{A}}$ of nonzero closed subspaces of a Banach space X and for which $\overline{sp}\{X_\alpha : \alpha \in \mathcal{A}\} = X$ is called a *functional unconditional decomposition* of X if for all $\alpha_1, \ldots, \alpha_n \in \mathcal{A}$, for all n, and for $x_j, x_j' \in X_{\alpha_j}$, where $\|x_j\| = \|x_j'\|$ for all j, it is true that $\|\sum_{j=1}^n x_j\| = \|\sum_{j=1}^n x_j'\|$. Further, we will say that a Banach space X is a *functional unconditional sum* of the family $(X_\alpha)_{\alpha\in\mathcal{A}}$ if there exists a normalized one-unconditional basis $\underline{u} = \{u_\alpha\}_{\alpha\in\mathcal{A}}$ for some Banach space E so that X is linearly isometric to $(\sum_{\mathcal{A}} \oplus X_\alpha)_{\underline{u}}$. The latter refers to the set of all $x = (x_\alpha)$ in the product space of the X_α's with $\sum_{\alpha\in\mathcal{A}} \|x_\alpha\| u_\alpha$ in E and norm given by $\|x\| = \|\sum_{\alpha\in\mathcal{A}} \|x_\alpha\| u_\alpha\|_E$. In the case in which X is real and each X_α is a real Hilbert space of dimension at least 2, we say that X is a *functional Hilbertian sum.* We will denote such spaces by FHS and say that "X is FHS." Also, we shall routinely abuse the notation by regarding X_α as a subspace of X when, in fact, it is isometric to such a subspace.

We observe that if $\overline{sp}\{X_\alpha : \alpha \in \mathcal{A}\} = X$, where the subspaces are well embedded, then X is a functional unconditional decomposition of the X_α if and only if X_α is orthogonal to X_β for all $\alpha \neq \beta$. Also, X is a functional unconditional sum of (X_α) if and only if there exists a functional unconditional decomposition (X_α') of X with X_α isometric to X_α' for all α.

If $\mathcal{H} = \{H_\lambda\}_{\lambda\in\Lambda}$ is the family of Hilbert components of Z with dim $H_\lambda \geq 2$, let FH(Z) denote the closed linear span of this family. We call FH(Z) the *functional Hilbertian part* of Z. By $\mathcal{O}r(Z)$ we will mean the set of all $x \in Z$ for which the closed linear span $sp\{x\}$ of x is either a Hilbert component of Z or $x = 0$. Note that FH(Z) corresponds to what we denoted by $\hat{h}(Z)$ in the complex case (except for the dimension requirement).

9.4.9. THEOREM. *(Rosenthal) If $X = FH(Z)$ then $\mathcal{H}$ is a functional unconditional decomposition of X, and $X = \overline{sp}\{TZ : T \in \mathfrak{A}(Z)$ with rank $T = 2\}$. For each $H \in \mathcal{H}$ and $x \in \mathcal{O}r(Z)$, $sp\{x\}$ is orthogonal to H. If X is* *orthogonally complemented in Z, then $\mathcal{O}(X) = \cap\{\mathcal{O}(H) : H \in \mathcal{H}\}$, and this holds if either of the following occur:*

(i) c_0 *does not embed in* X

or

(ii) $Z = \overline{sp}\{x : sp\{x\}$ *is orthogonally complemented in* $Z\}$.

PROOF. Let $H_1, H_2, \ldots, H_n$ (for $n \geq 1$) be distinct members of $\mathcal{H}$. By Theorem 9.4.8, the H_j's are pairwise orthogonal so that $H_j \subset \mathcal{O}(H_k)$ for $j \neq k$. For each $j = 1, \ldots, n$, let $x_j, x'_j \in H_j$ with $\|x_j\| = \|x'_j\|$ for each j. Because each H_j is well embedded, there exists for each j a surjective isometry U_j on Z so that $U_j x_j = x'_j$ and $U_j y = y$ for $y \in \mathcal{O}(H_j)$. Then $U = U_1 U_2 \cdots U_n \in \mathcal{G}(Z)$ and $U(\sum_{j=1}^n x_j) = \sum_{j=1}^n x'_j$. Therefore, $\|\sum_{j=1}^n x_j\| = \|\sum_{j=1}^n x'_j\|$, and we conclude that $\mathcal{H}$ is a functional unconditional decomposition of X.

To show that $X = \overline{sp}\{TZ : T \in \mathfrak{A}(Z)$ with *rank* $T = 2\}$, observe first that if $x \in H$ for some $H \in \mathcal{H}$, then because dim $H \geq 2$, there is a rank 2 skew Hermitian operator T on Z with $x \in TZ$. On the other hand, if $T \in \mathfrak{A}(Z)$ with rank $T = 2$, then by Proposition 9.4.5, TZ is a well-embedded two-dimensional Hilbert subspace of Z and so contained in a Hilbert component $H \in \mathcal{H}$.

If x is a nonzero element of $\mathcal{O}r(Z)$, then $sp\{x\}$ is a Hilbert component of Z distinct from each $H \in \mathcal{H}$ and so orthogonal to H by Theorem 9.4.8.

We omit the remainder of the proof. □

9.4.10. COROLLARY. *(Rosenthal) A real Banach space Z is a functional Hilbertian sum if and only if it equals the closed linear span of the ranges of its rank 2 skew-Hermitian operators.*

PROOF. If Z is FHS, then $FH(Z) = Z$ and equals the closed linear span of its rank 2 skew-Hermitians by the second statement of Theorem 9.4.9. Conversely, if the rank 2 condition holds, then $Z = FH(Z)$ by Theorem 9.4.9. As we observed earlier, a functional unconditional decomposition gives rise to a functional unconditional sum, which is therefore an FHS here. □

Note that if Z has a one-unconditional basis, then part (ii) of Theorem 9.4.9 is satisfied and $\mathcal{O}(X) = \overline{sp}(\mathcal{O}r(Z))$.

We next establish a result corresponding to Theorem 9.3.14.

9.4.11. THEOREM. *(Rosenthal) Let H be a Hilbert component of the real Banach space Z. Then UH is a Hilbert component for any surjective isometry U on Z, and if dim $H \geq 2$, $TH \subset H$ for all $T \in \mathfrak{A}(Z)$.*

PROOF. Since H is complemented, and U is surjective, we must have $\mathcal{O}(UH) = U\mathcal{O}(H)$. Furthermore, if W is a surjective isometry on UH, then for $u \in UH, v \in \mathcal{O}(UH)$,

$$\|Wu + v\| = \|WUx + Uy\| = \|U^{-1}WUx + y\| = \|x + y\| = \|u + v\|.$$

The equalities above have used the fact that H is well embedded and the conclusion is that UH is well embedded. Therefore, UH is itself a Hilbert component, for otherwise H would not be maximal.

Suppose now that T is skew-Hermitian and $x \in H$ with $\|x\| = 1$. The function f on $\mathbb{R}$ defined by $f(r) = \|x - e^{rT}x\|$ is continuous. Note that e^{rT} is an isometry and $e^{rT}x$ is in a Hilbert component $e^{rT}H$. If for a given r this Hilbert component is distinct from H, then we must have $e^{rT}H \subset \mathcal{O}(H)$ by Theorem 9.4.8. It now follows from the discussion just below Definition 9.4.3 that $f(r) = \|x - e^{rT}x\| \geq \|x\| = 1$. Since $\lim_{r \to 0} f(r) = 0$, there exists $\delta > 0$ such that $f(r) < 1$ for $|r| < \delta$, or equivalently, $e^{rT}x \in H$ for all such r. Since H is closed we can conclude that $d(e^{rT}x)/dr = Te^{rT}x \in H$ for $|r| < 1$. When $r = 0$ we have $Tx \in H$. □

9.4.12. Corollary. *(Rosenthal) Let $X = FH(Z)$ and $Y = \cap\{\mathcal{O}(H) : H \in \mathcal{H}\}$. Then X and Y are both invariant under $\mathcal{G}(Z)$ and $\mathfrak{A}(Z)$. Furthermore, if $U \in \mathcal{G}(Z)$, then $\mathcal{H} = \{UH : H \in \mathcal{H}\}$.*

Proof. First let $U \in \mathcal{G}(Z)$ and suppose $H \in \mathcal{H}$. Then $\dim UH > 2$ and UH is a Hilbert component by the theorem above. Thus $UX \subset X$, and since $U^{-1}X \subset X$, we actually have $UX = X$. Because U is surjective and $\mathcal{O}(UH) = U(\mathcal{O}(H))$, given $y \in Y$, we have $Uy \in \mathcal{O}(UH)$ for any $H \in \mathcal{H}$. Hence $UY \subset Y$, and as before, we actually have $UY = Y$. If $T \in \mathfrak{A}(Z)$ and $y \in Y$, it must be true that $e^{rT}y \in Y$ for all real r. Hence, since Y is closed, $d(e^{rT}y)/dr|_{r=0} = Ty \in Y$. The fact that $TX \subset X$ follows immediately from Theorem 9.4.11. □

It is staightforward to show that if $X = FH(Z)$ is orthogonally complemented and $Y = \mathcal{O}(X)$, then $\mathcal{G}(Z) = \mathcal{G}(X) \oplus \mathcal{G}(Y)$ precisely when X and Y are well embedded in Z. In this case we also have $\mathfrak{A}(Z) = \mathfrak{A}(X) \oplus \mathfrak{A}(Y)$.

Let us consider now a real Banach space E with a normalized one-unconditional basis $\{e_\alpha : \alpha \in \mathcal{A}\}$. Such a space is sometimes said to be *pure* if there are no rank 2 skew-Hermitian operators on E. We will show next that this meaning of the word *pure* coincides with the definition given in Section 9.2.

9.4.13. Theorem. *Let $\{e_\alpha : \alpha \in \mathcal{A}\}$ be a normalized one-unconditional basis for a real Banach space E. Then there are no rank 2 skew-Hermitian operators on E if and only if there are no equivalent coordinates with respect to the given basis. Furthermore, if $H \in \mathcal{H}(E)$, then $H = \overline{sp}\{e_\alpha : e_\alpha \in H\}$.*

Proof. Suppose $\alpha \sim \beta$. Then $sp\{e_\alpha, e_\beta\}$ is a well-embedded Hilbert subspace and so the range of a rank 2 skew-Hermitian operator $T = e_\alpha^* \otimes e_\beta - e_\beta^* \otimes e_\alpha$. On the other hand, suppose E has a rank 2 skew-Hermitian operator. Then by Proposition 9.4.5 there is a well-embedded Hilbert component H with $\dim H \geq 2$. For a given $\alpha \in \mathcal{A}$, either $e_\alpha \in \mathcal{O}r(E)$ so that its closed linear span is orthogonal to H, or there exists $H' \in \mathcal{H}(E)$ which contains $sp\{e_\alpha\}$. If $H' = H$, then $e_\alpha \in H$, or else $sp\{e_\alpha\}$ is again orthogonal to

H. It follows that there must exist $\beta \neq \alpha$ with $e_\alpha, e_\beta \in H$. Otherwise $\mathcal{O}(H) = E$ or $\mathcal{O}(H)$ is of co-dimension 1 in E. Neither of these is possible, so we must conclude that $sp\{e_\alpha, e_\beta\}$ is a well-embedded Hilbert subspace of E. (This is true since $sp\{e_\alpha, e_\beta\}$ is well embedded in H.) Note that $\mathcal{O}(sp\{e_\alpha, e_\beta\}) = \overline{sp}\{e_\gamma : \gamma \neq \alpha, \beta\}$. It is straightforward now to show that $\alpha \sim \beta$, where we use the fact that $\|z\| = \|\,|z|\,\|$ because the basis is one-unconditional. (Recall that $|z|$ denotes the element whose basis coefficients are the absolute values of those for z.)

As we noted above, if $H \in \mathcal{H}(E)$ and $\alpha \in \mathcal{A}$, then either $e_\alpha \in H$ or $sp\{e_\alpha\}$ is orthogonal to H. If $H' = \overline{sp}\{e_\alpha : \alpha \in \mathcal{A}, e_\alpha \in H\}$, then $\mathcal{O}(H') = \overline{sp}\{e_\beta : e_\beta \notin H\} \subset \mathcal{O}(H)$. Therefore, $H \subset H'$, so $H' = H$. □

A normalized one-unconditional basis for Z can be split into two pieces, one spanning $FH(Z)$ and the other being the part in $\mathcal{O}r(Z)$. If we select one basis element out of each member of $\mathcal{H}(Z)$, and combine these elements with the $\mathcal{O}r$ part, the result spans a pure space. If Z is an FHS space itself, then there is a one-unconditional basis $\underline{u} = \{u_\alpha\}_{\alpha \in \mathcal{A}}$ for a pure space E and Hilbert spaces $\{H_\alpha\}_{\alpha \in \mathcal{A}}$ with dim $H_\alpha \geq 2$ for all α such that Z is isometric to $(\sum_\alpha \oplus H_\alpha)_{\underline{u}}$. In other words, Z has a pure Hilbert space decomposition.

Let us now describe the Lie algebras of FHS spaces.

9.4.14. THEOREM. *(Rosenthal) Suppose that* $X = FH(Z)$ *is orthogonally complemented with* $Y = \mathcal{O}(FH(Z))$ *and let* $\mathcal{H}(Z) = \{H_\gamma : \gamma \in \Gamma\}$. *Suppose further that for each* $\gamma \in \Gamma, T_\gamma \in \mathfrak{A}(H_\gamma)$, *and* $M = \sup_\gamma \|T_\gamma\| < \infty$. *There is a unique* $T \in \mathfrak{A}(X)$ *with* $T|H_\gamma = T_\gamma$ *for each* $\gamma \in \Gamma$ *and* $Ty = 0$ *for* $y \in Y$*; moreover,* $\|T\| = M$. *Conversely, if* $T \in \mathfrak{A}(X)$, *then* $TH_\gamma \subset H_\gamma$ *for all* $\gamma \in \Gamma$ *and there is a* $\tilde{T} \in \mathfrak{A}(Z)$ *with* $\tilde{T}|X = T$ *and* $\tilde{T}(Y) = 0$. *Furthermore, if* $Z = X$, *then*

$$\mathfrak{A}(Z) = \left(\sum \oplus \mathfrak{A}(H_\gamma)\right)_{\ell^\infty(\Gamma)}.$$

PROOF. We are assuming that Z has a one-unconditional basis, and as we have observed above, we have a one-unconditional basis $\underline{u} = \{u_\gamma\}_{\gamma \in \Gamma}$ for a pure space E such that $X = (\sum_\gamma \oplus H_\gamma)_{\underline{u}}$. Given $z = x \oplus y$, where $x = \sum x_\gamma u_\gamma$, let us define T on Z to itself by

$$T(x \oplus y) = \sum_\gamma (T_\gamma x_\gamma) u_\gamma.$$

(Here we have abused the notation slightly, writing x as a "sum.") It is straightforward to show that T is a bounded operator with $\|T\| = M$. To show that $T \in \mathfrak{A}(Z)$, we show that $\exp(tT)$ is an isometry for each $t \in \mathbb{R}$.

Let n be a given positive integer with $\gamma_1, \ldots, \gamma_n \in \Gamma$. To simplify labeling, let $T_j = T_{\gamma_j}$ and $H_j = H_{\gamma_j}$. Now for each j we have $\exp(tT_j) \in \mathcal{G}(H_j)$. If we let $U_j = \exp(tT_j)$ on H_j and $U_j = I$ on $\mathcal{O}(H_j)$, we have

$$\|U_j(x_j \oplus y_j)\| = \|\exp(tT_j)(x_j) + y_j\| = \|x_j + y_j\|.$$

Since H_j is well embedded and T_j is skew-Hermitian, it follows that U_j is an isometry on $Z = H_j \oplus \mathcal{O}(H_j)$. Hence $U = U_1 \cdots U_n$ is also an isometry. Furthermore,

$$\begin{aligned} e^{tT}(x_1 + \cdots + x_n + y) &= e^{tT_1}x_1 + \cdots + e^{tT_n}x_n + y \\ &= U(x_1) + \cdots + U(x_n) + y \\ &= U(x_1 + \cdots + x_n + y). \end{aligned}$$

We conclude that $\exp(tT)$ must be an isometry since U is an isometry.

For the converse, if $T \in \mathfrak{A}(X)$, then $TH_\gamma \subset H_\gamma$ by Theorem 9.4.11. Thus $T_\gamma = T|H_\gamma$ has the properties as given in the first part, so we get $\tilde{T} \in \mathfrak{A}(Z)$ as desired.

The last statement should be clear. □

We remark, without providing proof, that a real Banach space E with one-unconditional basis that is also pure has a trivial Lie algebra; that is, $\mathfrak{A}(E) = \{0\}$. For a general Z with one-unconditional basis and for which $X = FH(Z)$ is orthogonally complemented, $\mathfrak{A}(Z) = \mathfrak{A}(X) \oplus \{0\}$. Our next theorem describes the isometries of an FHS space.

9.4.15. Theorem. *(Rosenthal) Let $\{H_\gamma\}_{\gamma\in\Gamma}$ be Hilbert spaces all of dimension at least 2, $\underline{u} = \{u_\gamma\}_{\gamma\in\Gamma}$ a one-unconditional basis for a pure space E, and $Z = (\sum_\Gamma \oplus H_\gamma)_{\underline{u}}$. Let $\mathcal{S}(Z)$ denote the set of all bijections σ on Γ which satisfy*

(i) *$\{u_{\sigma(\gamma)}\}_\Gamma$ is* isometrically equivalent *to $\{u_\gamma\}_\Gamma$; that is, there is a surjective isometry U on E for which $U(u_\gamma) = u_{\sigma(\gamma)}$ for each $\gamma \in \Gamma$, and*

(ii) *$H_{\sigma(\gamma)}$ is isometric to H_γ for all γ.*

Let $\sigma \in \mathcal{S}(Z)$. For each $\gamma \in \Gamma$, let $T_\gamma : H_\gamma \to H_{\sigma(\gamma)}$ be a surjective isometry. There is then a unique $T \in \mathcal{G}(Z)$ so that for all $x = (x_\gamma)_{\gamma\in\Gamma}$ in Z, $(Tx)_{\sigma(\gamma)} = Tx_\gamma$ for all $\gamma \in \Gamma$. Conversely, every surjective isometry T on Z is of this form.

Proof. Given $x = \sum_{\gamma\in\Gamma} x_\gamma$ with $x_\gamma \in H_\gamma$ for each γ, $\sum \|T_\gamma(x_\gamma)\| u_{\sigma(\gamma)}$ converges to an element of E. This is true, since for each finite $F \subset \Gamma$,

$$\left\| \sum_{\gamma\in F} \|T_\gamma(x_\gamma)\| u_{\sigma(\gamma)} \right\| = \left\| \sum_{\gamma\in F} \|x_\gamma\| u_\gamma \right\|.$$

Hence $Tx = \sum_{\gamma\in\Gamma} T_\gamma(x_\gamma)$ is a well- defined member of Z with $\|Tx\| = \|x\|$, so that T is a surjective isometry of Z.

Conversely, if T is a surjective isometry on Z, then $\mathcal{H}(Z) = \{H_\gamma : \gamma \in \Gamma\} = \{TH_\gamma : \gamma \in \Gamma\}$. It follows that for each $\gamma \in \Gamma$, there is a unique $\sigma(\gamma)$ such that $TH_\gamma = H_{\sigma(\gamma)}$. If $T_\gamma = T|H_\gamma$, then T_γ is a surjective isometry from H_γ onto $H_{\sigma(\gamma)}$. Clearly σ is a bijection. For each $\gamma \in \Gamma$, let $h_\gamma \in H_\gamma$ with $\|h_\gamma\| = 1$. It can be shown that $\{h_\gamma\}_\Gamma$ forms a basis for a pure space which is isometrically equivalent to $\{u_\gamma\}_\Gamma$ and $\{Th_\gamma\}_\Gamma$ is isometrically equivalent to $\{u_{\sigma(\gamma)}\}_\Gamma$. Since T is an isometry, $\{h_\gamma\}$ and $\{Th_\gamma\}$ are isometrically equivalent. $\square$

We will close this section by looking at some examples. Let a norm on $\mathbb{R}^3$ be defined by

$$\|a_1e_1 + a_2e_2 + a_3e_3\| = \max\left\{\sqrt{a_1^2 + a_2^2}, \sqrt{a_2^2 + a_3^2}, \sqrt{a_1^2 + a_3^2}\right\},$$

where $\{e_1, e_2, e_3\}$ is the usual orthonormal basis. For any pair e_j, e_k with $j \neq k$, $sp\{e_j, e_k\}$ is an orthogonally complemented Euclidean subspace which is not well embedded. For example, $sp\{e_1, e_2\}$ has orthogonal complement $sp\{e_3\}$ and

$$\left\|e_1 + \frac{1}{\sqrt{2}}e_3\right\| = \sqrt{\frac{3}{2}}$$

while

$$\left\|\frac{e_1 + e_2}{\sqrt{2}} + \frac{1}{\sqrt{2}}e_3\right\| = 1.$$

Thus $E = (R^3, \|\cdot\|)$ is a pure space. Note that this is true even though E has two-dimensional Hilbert subspaces.

If we define a norm $\|\cdot\|'$ on $\mathbb{R}^4$ by

$$\|(a_1, a_2, a_3, a_4)\|' = \|(\sqrt{a_1^2 + a_2^2})e_1 + a_3e_2 + a_4e_3\|,$$

then $Z = (\mathbb{R}^4, \|\cdot\|')$ is a real Banach space with normalized one-unconditional basis $\{e_1, e_2, e_3, e_4\}$. It can be easily seen that $X = sp\{e_1, e_2\}$ is a well-embedded Euclidean subspace of Z which is a Hilbert component. To see this last statement, it suffices to show, by Theorem 9.4.13, that $sp\{e_1, e_2, e_j\}$ is not well embedded for $j = 3$ or 4. When $j = 3$, we see this by considering the fact that $\sqrt{3} = \|(1, 1, 0, 1)\|' \neq \|(0, 1, 1, 1)\|' = \sqrt{2}$. Thus $X = FH(Z)$, and Z can be identified with the E direct sum of the Hilbert components X, $sp\{e_3\}$, and $sp\{e_4\}$, where E is the pure space of the previous paragraph. The Lie algebra of Z consists of the operators of the form $T \oplus 0$ where T is skew-symmetric on R^2. The set $\mathcal{S}(Z)$ of Theorem 9.4.15 is the set of permutations on $\{1, 2, 3, 4\}$ that preserve the sets $\{1, 2\}$ and $\{3, 4\}$. The isometries are of the form $U \oplus V$, where U is an isometry of the two-dimensional Euclidean space and V is a permutation isometry on $Y = sp\{e_3, e_4\}$.

9.5. Decompositions with Banach Space Factors

In the previous sections, the decomposition of the given space always resulted in a direct sum of Hilbert spaces. In this section we follow a method introduced by Li and Randrianantoanina that leads to direct sums whose factors may be Banach spaces which are not Hilbert spaces. We will assume that X is a space with a normalized one-unconditional basis $\{e_j\}_{j\in N}$ (or equivalently, X is an admissible sequence space) where $N = \mathbb{N}$ or $\{1, 2, \ldots, n\}$

for some $n \in \mathbb{N}$. The goal is to write $X = E(X_j)_{j \in N}$, where $E, X_1, X_2, \ldots$ are sequence spaces. The key to this approach is the notion of a *fiber*, which can be regarded as a generalization of the concepts of equivalent coordinates or well-embedded subspaces.

9.5.1. Definition. *Let (X, ν) be an admissible sequence space with one-unconditional basis $\{e_j\}_{j \in N}$. A nonempty proper subset S of N is said to be a* fiber *if for all finitely nonzero sequences $\{a_s\}_{s \in S}, \{a'_s\}_{s \in S}$ of scalars such that*

$$\nu\left(\sum_{s \in S} a_s e_s\right) = \nu\left(\sum_{s \in S} a'_s e_s\right),$$

then

$$\nu\left(\sum_{s \in S} a_s e_s + \sum_{j \notin S} b_j e_j\right) = \nu\left(\sum_{s \in S} a'_s e_s + \sum_{j \notin S} b_j e_j\right)$$

for all finitely nonzero sequences of scalars $\{b_j\}_{j \in N \setminus S}$. The corresponding fiber space *is given by*

$$X_S = \overline{sp}\{e_s : s \in S\}.$$

Clearly, any singleton set is a fiber, and every proper subset of N for an ℓ^p space is a fiber. For the space

$$X = \ell^1(2) \oplus_1 \ell^2(2) \oplus_1 \ell^\infty(2), \tag{75}$$

the fibers, in addition to the singletons, are

$$\{1,2\},\ \{3,4\},\ \{5,6\},\ \{3,4,5,6\},$$
$$\{1,2,3,4\},\ \{1,2,5,6\},\ \{1,3,4,5,6\},\ \{2,3,4,5,6\}.$$

We observe that the last four fibers listed above are maximal fibers, in that they are properly contained in no other fiber. Maximal fibers will play an important role in what follows, but not every space has maximal fibers. Consider the space $X_{p,q}$ with the norm defined inductively as follows:

$$\nu(x_1 e_1) = |x_1|,$$

$$\nu\left(\sum_{j=1}^{n} x_j e_j\right) = \left(\left(\nu\left(\sum_{j=1}^{n-1} x_j e_j\right)\right)^q + |x_n|^q\right)^{1/q} \quad \text{if } n \geq 2 \text{ is odd}$$

$$= \left(\left(\nu\left(\sum_{j=1}^{n-1} x_j e_j\right)\right)^p + |x_n|^p\right)^{1/p} \quad \text{if } n \geq 2 \text{ is even},$$

where $1 \leq p, q < \infty$ and $p \neq q$. Here, the fibers are of the form $\{1, 2, \ldots, m\}$ for $m \in \mathbb{N}$, and so in the infinite-dimensional case, there are no maximal fibers.

The reader will have noticed that if S is a fiber, then the fiber space X_S is well embedded in the language of the previous section. Also, if $j \sim k$ as defined in Section 9.2, then $\{j,k\}$ would be a fiber which we will now call an ℓ^2 fiber. We make use of this idea to generalize the notion of equivalent coordinates.

9.5.2. DEFINITION. *Two coordinates j,k are said to be* norm equivalent, *written $j \sim_\nu k$, if $\{j,k\}$ is a fiber.*

At this point we should show that the notion of norm equivalent is, indeed, an equivalence relation. Let us assume for a moment, that for any pair j,k of indices and nonnegative scalars, it is true that

$$\nu(se_j + te_k) = \nu(te_j + se_k). \tag{76}$$

If we suppose that $j \sim_\nu k$ and $k \sim_\nu m$, we note that if s,t are scalars, then

$$\nu(se_j + te_k) = \nu(se_j + te_m) = \nu(se_k + te_m), \tag{77}$$

where the first equality is true because $k \sim_\nu m$ and the second follows from $j \sim_\nu k$. Now suppose $\nu(se_j + re_m) = \nu(s'e_j + r'e_m)$. To show that $\{j,m\}$ is a fiber, it suffices to show, for any scalar t and any u which is a finite linear combination of basis vectors with indices different from j,k,m, that

$$\nu(se_j + te_k + re_m + u) = \nu(s'e_j + te_k + r'e_m + u). \tag{78}$$

For s,r,s',r' as given and for any $t>0$ we obtain

$$\begin{aligned}
\nu(se_j + te_k + re_m + u) &= \nu(\nu(se_j + te_k)e_k + re_m + u) \quad (j \sim_\nu k)\\
&= \nu(te_j + se_k)e_k + re_m + u) \quad (\text{by } (76))\\
&= \nu(te_j + se_k + re_m + u) \quad (j \sim_\nu k)\\
&= \nu(te_j + \nu(se_k + re_m)e_m + u) \quad (k \sim_\nu m)\\
&= \nu(te_j + \nu(se_j + re_m)e_m + u) \quad (\text{from } (77))\\
&= \nu(te_j + \nu(s'e_j + r'e_m)e_m + u)\\
&= \nu(s'e_j + te_k + r'e_m + u),
\end{aligned}$$

where the last equality comes from reversing the previous steps. Thus (78) is satisfied and we would have established the transitive property for the relation $\sim_\nu$. Although it is not readily apparent that (76) is satisfied, we will see, because of a lovely old theorem of Bohnenblust, that in the situation above, the norm is actually an ℓ^p norm.

We are going to state and sketch the proof of a slightly stronger form of Bohnenblust's theorem. The strengthening of the theorem is due to Randrianantoanina.

9.5.3. THEOREM. *(Bohnenblust and Randrianantoanina) Let f be a real-valued function of two nonnegative variables which satisfies*

(i) $f(cs,ct) = cf(s,t)$,
(ii) $f(s,t) \le f(s',t')$ *for* $0 \le s \le s'$, $0 \le t \le t'$,

(iii) $f(0,1) = f(1,0) = 1$,
(iv) $f(f(s,t),r) = f(s,f(t,r))$ *for* $s,t,r \geq 0$.

Then there exists p*, with* $0 < p \leq \infty$ *such that*

$$f(s,t) = (s^p + t^p)^{1/p},$$

where in the case $p = \infty$*, we mean* $f(s,t) = \max\{s,t\}$.

PROOF. Such a function f gives rise to a nondecreasing sequence $\{a_n\}$ of real numbers defined inductively by

$$a_1 = 1; \quad a_n = f(1, a_{n-1}) \text{ for } n > 1.$$

There are four steps in the proof, namely showing: (1) $a_{n+m} = f(a_n, a_m)$; (2) $a_n a_m = a_{nm}$; (3) $a_2 = 1$ implies $f(s,t) = \max\{s,t\}$; (4) $a_2 > 1$ implies $a_n = n^{1/p}$ for some real number $p > 0$, and $f(1, r^{1/p}) = (1+r)^{1/p}$ for any $r > 0$. We begin with step (1).

Our first claim is that $f(1,a_n) = f(a_n,1)$ for each n. The statement is trivial for $n = 1$, and if it holds for $n-1$, we have

$$\begin{aligned} f(1,a_n) &= f(1, f(1,a_{n-1})) \\ &= f(1, f(a_{n-1},1)) \\ &= f(f(1,a_{n-1}),1) \quad \text{(by (iv))} \\ &= f(a_n, 1) \end{aligned}$$

so the claim is established by induction. Now for any n, m, we can use the claim and ((iv)) repeatedly to obtain

$$\begin{aligned} f(a_n,a_m) &= f(f(1,a_{n-1}),a_m) \\ &= f(f(a_{n-1},1),a_m) = f(a_{n-1}, f(1,a_m)) \\ &= f(a_{n-1},a_{m+1}) = f(a_{n-2},a_{m+2}) = \dots \\ &\qquad\qquad = f(a_1, a_{m+n-1}) = a_{m+n}, \end{aligned}$$

which completes step (1).

For step (2), we assume n is fixed and induct on m. The statement is again trivially true for $m = 1$. We assume true for $m-1$. Then

$$\begin{aligned} a_n a_m = a_n f(1,a_{m-1}) &= f(a_n, a_n a_{m-1}) \text{ by (i)} \\ &= f(a_n, a_{n(m-1)}) \quad \text{induction hyp} \\ &= a_{nm} \quad \text{by step (1).} \end{aligned}$$

Next suppose $a_2 = 1$. Repeated application of the result of step (2) leads to $a_{2^n} = 1$, and therefore, $a_n = 1$ for every n. Given any t with $0 \leq t \leq 1$,

$$1 = f(1,0) \leq f(1,t) \leq f(1,1) = 1,$$

so that $f(1,t) = 1$. From this and condition (iii) we conclude that

$$f(s,t) = max\{s,t\},$$

and step (3) is finished.

For the final step, let m, n be fixed positive integers greater than 1 and for a given positive integer k let h be such that

$$m^h \leq n^k < m^{h+1}.$$

From step (2) we obtain $a_m^h = a_{m^h}$, and with similar statements for k and $h+1$ we see that

$$h \log m \leq k \log n < (h+1) \log m$$

and

$$h \log a_m \leq k \log a_n < (h+1) \log a_m.$$

From these inequalities, we have

$$\begin{aligned} \log a_n &\leq \frac{h}{k} \log a_m + \frac{1}{k} \log a_m \\ &\leq \frac{\log n}{\log m} \log \alpha_m + \frac{1}{k} \log a_m. \end{aligned}$$

Since this holds for all k, we conclude that

$$\frac{\log a_n}{\log n} \leq \frac{\log a_m}{\log m},$$

and upon interchanging the roles of n, m, we have

$$\frac{\log a_n}{\log n} = \frac{\log \alpha_m}{\log m}.$$

Hence, $\frac{\log a_n}{\log n}$ is independent of n, so we may let p be defined by

$$\frac{\log a_n}{\log n} = \frac{1}{p}$$

for any n. It follows that $a_n = n^{1/p}$. (Here, of course, we are assuming that $a_2 > 1$.) If we now put $r = m/n$, we must have

$$\begin{aligned} f(1, m^{1/p}/n^{1/p}) = n^{-1/p} f(n^{1/p}, m^{1/p}) &= n^{-1/p} f(a_n, a_m) \\ = n^{-1/p} a_{n+m} &= n^{-1/p}(n+m)^{1/p} = (1+r)^{1/p}. \end{aligned}$$

We extend this to $f(1, r^{1/p}) = (1+r)^{1/p}$ for all real r and the desired result follows by property (i). □

We are now in a position to show that $\sim_\nu$ is an equivalence relation, and what is more, the fiber space corresponding to an equivalence class with more than two elements must be an ℓ^p space for some p.

9.5.4. PROPOSITION. *The relation $\sim_\nu$ given in Definition 9.5.2 is an equivalence relation. Furthermore, if N_j is an equivalence class with more than two elements, then the corresponding fiber space X_j is an ℓ^p space for some p, $1 \leq p \leq \infty$.*

PROOF. Again, the only thing in question about equivalence is the transitive property. Hence, suppose that $j \sim_\nu k$ and $k \sim_\nu m$, where j, k, m are all distinct. For any nonnegative numbers s, t, r, we note that

$$\begin{aligned}(79)\qquad \nu(se_j + te_k + re_m) &= \nu(\nu(se_j + te_k)e_j + re_m) \text{ (since } j \sim_\nu k) \\ &= \nu(\nu(se_j + te_k)e_j + re_k) \text{ by (77).}\end{aligned}$$

Also,

$$\begin{aligned}(80)\qquad \nu(se_j + te_k + re_m) &= \nu(se_j + \nu(te_k + re_m)e_k) \text{ (since } k \sim_\nu m) \\ &= \nu(se_j + \nu(te_j) + re_k)e_k) \text{ (by (77)).}\end{aligned}$$

If we define $f(s,t) = \nu(se_j + te_k)$, then f clearly satisfies conditions (i), (ii), and (iii) of Bohnenblust's theorem. The fact that (iv) holds is a consequence of (79) and (80). Hence, by Theorem 9.5.3, there exists p such that $\nu(se_j+te_k) = (|s|^p + |t|^p)^{1/p}$ or $\max\{|s|, |t|\}$ if $p = \infty$. In particular, equation (76) holds and the argument given just prior to the statement of Bohnenblust's theorem shows that $j \sim_\nu m$. If j, k, m are not all distinct, there is nothing to prove, so we see that transitivity holds. It is clear, by induction, that the norm of any finite linear combination of basis vectors in X_j, given that there are at least three, is just the ℓ^p norm, and this result extends by continuity of the norm to an infinite sum.

□

Let $\mathcal{M} = \{N_j : j \in \Gamma\}$ denote the equivalence classes determined by the equivalence relation given in Definition 9.5.2. Here, Γ is of the form $\{1, 2, \ldots, k\}$, or $\Gamma = \mathbb{N}$. By X_j we will mean the closed linear span of the basis vectors e_m for $m \in N_j$.

9.5.5. THEOREM. *Let (X, ν) denote a sequence space with normalized absolute norm. Then there exists a sequence space (E, μ) whose coordinate vectors form a normalized one-unconditional basis such that $X = E((X_j)_{j\in\Gamma})$, where the X_j's are as described just prior to the statement of the theorem. Furthermore, for each X_j of dimension greater than 2, there is some p, $1 \leq p \leq \infty$, such that $X_j = \ell^p$ (where ℓ^p is of the appropriate dimension).*

PROOF. The proof follows just as in the proof of Theorem 9.2.4. As we did there, if the elements of N_j are denoted by $p_{j1}, p_{j2}, \ldots$, we can let E denote the space of sequences $\alpha = (\alpha(j))$ for which $\sum_j \alpha(j)e_{p_{j1}} \in X$ and the norm is defined by $\mu(\alpha) = \nu(\sum_j \alpha(j)e_{p_{j1}})$. □

9.5.6. REMARKS.

(i) *The original statement of Bohnenblust's theorem (Theorem 9.5.3) required one more condition on f; namely that $f(s,t) = f(t,s)$. That condition has been removed. More will be said about this in the notes at the end of the chapter.*

(ii) *We observe that in contrast to the situation in the previous sections, the fiber spaces X_j resulting from the equivalence classes can be ℓ^p*

spaces for different p's. The Hilbert spaces that arise in the earlier decompositions will still show up, but other non-one-dimensional spaces may occur as well. In the example given in (75), there are three two-element equivalence classes, and the fiber spaces are $\ell^1(2), \ell^2(2)$, and $\ell^\infty(2)$. The space we called $X_{p,q}$ has one two-dimensional summand, $\ell^2(2)$, and the others are one-dimensional. The space E in this case is not an ℓ^p space, but is $X_{q,p}$ instead.

(iii) *Unlike the situation in Theorem 9.2.4, the space E in the decomposition of Theorem 9.5.5 can certainly have norm-equivalent coordinates.*

(iv) *One could define an ℓ^p equivalence in analogy to the equivalence relation defined in Section 9.2, and then the fiber spaces would all be ℓ^p spaces. In one sense, this is as good as using ℓ^2 fibers. On the other hand, the corresponding decomposition would not be as compatible with describing Hermitian operators and using them in finding isometries. The $\sim_\nu$ relation seems most natural as far as the given norm is concerned.*

(v) *There can certainly be summands that are not ℓ^p spaces for any p. This can happen when equivalence classes have only two elements. It can happen that $\nu(se_j + te_k) \neq \nu(te_j + se_k)$.*

In connection with the last comment above, let us consider a couple of examples. Let ν_0 be a norm defined on $\mathbb{C}^2$ (or $\mathbb{R}^2$) by

$$\nu_0(s,t) = \begin{cases} |t|, & \text{if } 4|s| \le |t|; \\ |s| + \frac{3}{4}|t|, & \text{if } 4|s| > |t|. \end{cases} \tag{81}$$

Note that $\nu_0(1/4, 1) = 1 < \nu_0(1, 1/4)$. If we define ν_1 on $\mathbb{C}^3$ by

$$\nu_1(s,t,r) = \nu_0(|s| + \frac{4}{7}|t|, \frac{4}{7}|t| + |r|), \tag{82}$$

we get a three-dimensional space with no equivalent coordinates and which does not satisfy the symmetry condition given by (76). On the other hand, the norm ν_2 defined on $\mathbb{C}^3$ by

$$\nu_2(s,t,r) = \nu_0(|s| + |t|, |r|) \tag{83}$$

gives a three-dimensional space in which the symmetry condition is not always satisfied, and there is one pair of equivalent coordinates, $1 \sim_\nu 2$. The corresponding fiber space is $\ell^1(2)$. Various direct sums involving these norms also give interesting examples.

We now describe the isometries on these decompositions. We find it necessary to assume our spaces are complex. We will see that the theorem does not carry over directly to the real case.

9.5.7. THEOREM. *Let X be a complex sequence space with normalized one-unconditional basis $\{e_j : j \in N\}$ and suppose $X = E((X_j)_{j\in\Gamma})$ as given in Theorem 9.5.5. Then T is an isometry on X if and only if there exists* *a permutation* π *in the symmetry class of the norm* μ *and a family* $\{T_j\}$ *of surjective isometries with* $T_j : X_{\pi(j)} \to X_j$ *such that*

$$T(\sum_{j\in\Gamma} x_j) = \sum_{j\in\Gamma} T_j x_{\pi(j)}, \tag{84}$$

where we write $x = \sum_{j\in\Gamma} x_j$ *for an element of the decomposition.*

Proof. Let $\mathcal{M}_2 = \{J_\lambda : \lambda \in \Lambda\}$ denote the collection of maximal ℓ^2 fibers, that is, the collection of equivalence classes determined by the equivalence relation $\sim$ of the previous sections. If T is an isometry, then by Theorem 9.2.12 or Theorem 9.3.14 there exists a permutation σ of Λ such that for all $\lambda \in \Lambda$,

$$\text{supp }(T(Y_\lambda)) = J_{\sigma(\lambda)} \tag{85}$$

where by *supp* we mean the indices of the basis elements for the given space and $Y_\lambda = \overline{sp}\{e_j : j \in J_\lambda\}$. If $A \in \mathcal{M}$, then $A = \cup_{\lambda\in\Lambda_A} J_\lambda$, where the union consists of one element, A itself, if $A \in \mathcal{M}_2$, and a union of singletons otherwise. For each $A = \cup_{\lambda\in\Lambda_A} J_\lambda$, we define $\widetilde{T}$ by

$$\widetilde{T}(A) = \cup_{\lambda\in\Lambda_A} J_{\sigma(\lambda)}.$$

In case A is itself an ℓ_2 fiber, it is clear that $\widetilde{T}(A)$ is in $\mathcal{M}_2$ (also in $\mathcal{M}$) and of the same cardinality as A. If A is a union of singletons, then $\widetilde{T}(A) = \cup_{\lambda\in\Lambda_A} J_{\sigma(\lambda)}$ where each $J_{\sigma(\lambda)}$ is also a singleton.

Let $j, k \in \widetilde{T}(A)$, so that $j \in J_{\sigma(\lambda_1)}, k \in J_{\sigma(\lambda_2)}$, where $\lambda_1 \neq \lambda_2$. Let $a, b \in \overline{sp}\{e_j, e_k\}$ with $\nu(a) = \nu(b)$. Further, suppose $c \in X$ with supp c contained in $N \setminus \{j, k\}$. Note that $T^{-1}(a)$ and $T^{-1}(b)$ have supports that are singletons and are in A, so they are norm equivalent, while supp $T^{-1}(c)$ is disjoint from both. Hence

$$\nu(a + c) = \nu(T^{-1}(a) + T^{-1}(c)) = \nu(T^{-1}(b) + T^{-1}(c)) = \nu(b + c),$$

from which we conclude that $j \sim_\nu k$ and $\widetilde{T}(A)$ is also in $\mathcal{M}$. Therefore, for each $N_j \in \mathcal{M}$, there is some $\pi(j) \in \Gamma$ such that

$$\widetilde{T}(N_{\pi(j)}) = N_j.$$

Moreover, this means that T must map $X_{\pi(j)}$ onto X_j and its restriction to $X_{\pi(j)}$ is an isometry. □

We see that the proof of our theorem really depends on the corresponding result for Hilbert space decompositions, and in general, there is not such a comprehensive theorem in the real case. If the space is an FHS space, we do get the needed permutation σ (by Theorem 9.4.15), but in this case, there would be no norm equivalence classes that are not ℓ^2 fibers and so no non-Hilbert spaces in the decomposition. If we write the space in the form $X \oplus Y$, where X includes the part generated by the ℓ^2 fibers, then isometries map X and Y back to themselves (Corollary 9.4.12), but we cannot, in general, be sure whether the restriction to Y has the form of (85). For example, let X be the real space

$$X = \ell^1(3) \oplus_1 \ell^\infty(2). \tag{86}$$

The operator defined by

$$T(x(1), x(2), x(3), x(4), x(5)) = (U(x(4), x(5)), x(3), U^{-1}(x(1), x(2))),$$

where $U(s,t) = 2^{-1}(s+t, s-t)$, is an isometry which does not have the form of (85), since the norm equivalence classes are $\{1,2,3\}$ and $\{4,5\}$. The problem here is that in the real case, $\ell^\infty(2)$ is isometric with $\ell^1(2)$ under the map U.

For the remainder of this section, we will assume that the space X has maximal fibers. This will allow for a possibly different and sometimes more useful description of the decompositions.

9.5.8. PROPOSITION. *(Li and Randrianantoanina) Let X be a sequence space with normalized one-unconditional basis $\{e_j : j \in N\}$. Suppose there exist two maximal fibers S and T such that $S \cap T \neq \emptyset$. Then $S \cup T = N$ and*

$$X = X_{T\setminus S} \oplus_p X_{T\cap S} \oplus_p X_{S\setminus T} = \ell^p(X_{T\setminus S}, X_{T\cap S}, X_{S\setminus T}),$$

for some $1 \leq p \leq \infty$.

PROOF. First of all, it can be shown, since S and T are fibers sharing a common element, that $S \cup T$ is also a fiber. We omit those details. By maximality of S, it follows that $S \cup T = N$. Furthermore, since both S and T are maximal, we have $S \setminus T$ and $T \setminus S$ are both nonempty. Let $j \in S \setminus T$, $k \in S \cap T$, and $m \in T \setminus S$ be given. Let $f(s,t) = \nu(se_j + te_k)$. Then, in exactly the same way as in the proof of Proposition 9.5.4, we can show that f satisfies the hypotheses of Bohnenblust's theorem. Thus,

$$\nu(se_j + te_k) = \begin{cases} (|s|^p + |t|^p)^{1/p} & \text{if } p < \infty; \\ \max(|s|, |t|) & \text{if } p = \infty. \end{cases}$$

Hence, for all scalars s, t, r,

$$\nu(se_j + te_k + re_m) = \ell^p(s, t, r).$$

Now, if $x = (x(i))$ is a sequence with finitely many nonzero entries, we use the fact that S and T are fibers to get

$$\begin{aligned} \nu\left(\sum_{i\in T} x(i)e_i\right) &= \nu\left(\nu\left(\sum_{i\in S\cap T} x(i)e_i\right)e_j + \nu\left(\sum_{i\in T\setminus S} x(i)e_i\right)e_m\right) \\ &= \ell^p\left(\nu\left(\sum_{i\in S\cap T} x(i)e_i\right), \nu\left(\sum_{i\in T\setminus S} x(i)e_i\right)\right). \end{aligned}$$

Also, since $S \cup T = N$, we have

$$\nu\left(\sum_{i\in N} x(i)e_i\right) = \nu\left(\nu\left(\sum_{i\in T} x(i)e_i\right)e_m + \nu\left(\sum_{i\in S\setminus T} x(i)e_i\right)e_j\right)$$
$$= \ell^p\left(\nu\left(\sum_{i\in T\setminus S} x(i)e_i\right), \nu\left(\sum_{i\in T\cap S} x(i)e_i\right), \nu\left(\sum_{i\in S\setminus T} x(i)e_i\right)\right).$$

Since finitely supported elements are dense and the norm is continuous, we obtain the desired result. □

We want to show now how maximal fibers can be used to determine a direct sum structure that is useful and may be different than what we have previously obtained using the norm equivalence classes. In this description it is useful to identify a certain type of subspace that can occur in the real case. (Recall the example given by equation (86) above.)

9.5.9. Definition. *For $1 \le p < \infty$, $p \ne 2$, let $E_p(2)$ denote the space $\mathbb{R}^2$ with the norm given by*

$$\|(s,t)\|_{E_p} = \left(\frac{|s+t|^p}{2} + \frac{|s-t|^p}{2}\right)^{1/p}.$$

If $p = \infty$, let

$$\|(s,t)\|_{E_\infty} = \max(|s+t|, |s-t|) = \|(s,t)\|_{\ell^1}.$$

Note that $E_p(2)$ is isometric to $\ell^p(2)$ by means of the isometry $U(s,t) = 2^{-1/p}(s+t, s-t)$. Also, $E_p(2)$ can be written as an ℓ^p sum of two nonzero subspaces.

9.5.10. Theorem. *(Li and Randrianantoanina) Let X be a real or complex sequence space with normalized one-unconditional basis $\{e_j : j \in N\}$. Suppose that $X \ne \ell^p(N)$ for any $1 \le p \le \infty$ and that X has maximal fibers. Then there exists a subset Γ of $\mathbb{N}$ with two or more elements (and of the form $\Gamma = \{1, 2, \ldots, k\}$, or $\Gamma = \mathbb{N}$), a collection $\{S_j : j \in \Gamma\}$ of subsets of N, with $\emptyset \subseteq S_1$ and such that $\{S_j : j \ge 2\}$ forms a partition of $N \setminus S_1$, and X is a direct sum of the subspaces $X_j = \overline{sp}\{e_i : i \in S_j\}$. Furthermore, exactly one of the following holds:*

(i) *for each $j \in \Gamma$, S_j is a maximal fiber and $X = E((X_j)_{j\in\Gamma})$ where the norm μ on E is defined by*

$$\mu((a_j)_{j\in\Gamma}) = \nu\left(\sum_{j\in\Gamma} a_j e_{j_1}\right),$$

where $j_1 \in S_j$. In this case maximal fibers of E exist and are singletons;

(ii) *for each $j \in \Gamma$ with $j \ge 2$, $N \setminus S_j$ is a maximal fiber and there exists p with $1 \le p \le \infty$ such that $X = \ell^p((X_j)_{j\in\Gamma})$, where*
   (a) *$X_1 = \ell^p(\text{card}(S_1))$ (possibly dim $X_1 = 0$),*

(b) *some of the X_j's equal $E_p(2)$ if $\mathbb{F} = \mathbb{R}$ and $p \neq 2$,*
(c) *and the rest of the X_j's are such that dim $X_j \geq 2$ and X_j is not an ℓ^p sum of two nonzero subspaces.*

PROOF. (i) Suppose first that each S_j is maximal and we have $X = E((X_j)_{j\in\Gamma})$ with norm as given in (i) above. If $F \subset \Gamma$ is a fiber in E, it can be shown that $S_F = \cup_{j\in F} S_j$ is a fiber in X. This follows by using, in turn, the facts that each S_j is a fiber, and that F is a fiber in E. We omit the details. Since each S_j is a maximal fiber, we conclude that F must be a singleton.

(ii) If we suppose now it is not possible to form a partition of N consisting of maximal fibers, there must exist two maximal fibers whose intersection is nonempty. By Proposition 9.5.8 there is some p, $1 \leq p \leq \infty$, and spaces Y_1, Y_2, Y_3 corresponding to a partition A_1, A_2, A_3 of N so that $X = \ell^p(Y_1, Y_2, Y_3)$. Among all decompositions of X into an ℓ^p sum, let $\{R_k : k \in \Lambda\}$ be a partition of N that is maximal and such that $X = \ell^p((Z_k)_{k\in\Lambda})$, where the support of the basis elements for Z_k is R_k. If X is real and $p \neq 2$, then for each k we have one of the three possibilities: (a) R_k is a singleton; (b) R_k has two elements and $Z_k = E_p(2)$; or (c) R_k has at least two elements and Z_k cannot be decomposed as an ℓ^p direct sum of two nonzero subspaces. If X is complex, only (a) and (c) can happen.

Let S_1 be the union of the R_k which are singletons, and rename the other R_k by S_j as required. It follows that condition (ii) of the statement of the theorem holds. □

9.5.11. REMARKS. *Let $\mathcal{M}$ denote the collection of all maximal fibers for X.*

(i) *When there is a partition of N by maximal fibers, $\mathcal{M}$ consists exactly of the elements of the partition.*
(ii) *Thus, in case (i) of the theorem above, $\mathcal{M} = \{S_j : j \in \Gamma\}$.*
(iii) *In case (ii), $\mathcal{M} \neq \{N \setminus S_j : j \geq 2\}$ only if $S_1 \neq \emptyset$. In fact, if $S_1 \neq \emptyset$, then*

$$\mathcal{M} = \{N \setminus \{k\} : k \in S_1\} \cup \{N \setminus S_j\}.$$

Finally, we want to show that the conclusion of Theorem 9.5.7 also holds for the decomposition given above. Again, we must stick to the complex case.

9.5.12. THEOREM. *(Li and Randrianantoanina) Let X be a complex sequence space with normalized one-unconditional basis $\{e_j : j \in N\}$. Assume also that X has maximal fibers so that $X = E((X_j)_{j\in\Gamma})$ as given in Theorem 9.5.10. Then T is an isometry on X if and only if there exists a permutation π in the symmetry class of the norm μ and a family $\{T_j\}$ of surjective isometries with $T_j : X_{\pi(j)} \to X_j$ such that*

$$T(\sum_{j\in\Gamma} x_j) = \sum_{j\in\Gamma} T_j x_{\pi(j)}, \tag{87}$$

where we write $x = \sum_{j\in\Gamma} x_j$ for an element of the decomposition.

PROOF. We follow the same pattern and notation as in the proof of Theorem 9.5.7 except now we let $\mathcal{M}$ be the family of maximal fibers. First we show that

$$\widetilde{T}(\mathcal{M} \cap \mathcal{U}) = \mathcal{M} \cap \mathcal{U}, \tag{88}$$

where $\mathcal{U}$ is the collection of subsets of Γ which are unions of elements from $\mathcal{M}_2$. To see this, suppose $S \in \mathcal{M} \cap \mathcal{U}$ and observe that $S^c \in \mathcal{U}$. The proof that $\widetilde{T}(S)$ is a fiber goes in essentially the same way as the proof that $\widetilde{T}$ preserves equivalence classes in the earlier theorem. Assume that $\widetilde{T}(S) \notin \mathcal{M}$, so that $\widetilde{T}(S)$ is a subfiber of a proper fiber R. Then $\widetilde{T^{-1}}(R)$ is a proper fiber in X that contains $(\widetilde{T^{-1}})(\widetilde{T}(S) = S$, which contradicts the maximality of S. Therefore, $\widetilde{T}(\mathcal{M} \cap \mathcal{U}) \subseteq \mathcal{M} \cap \mathcal{U}$, and since the same subset inequality holds for $\widetilde{T^{-1}}$, we conclude that (88) holds.

Next, we observe that

$$\mathcal{M} \subset \mathcal{U} \tag{89}$$

unless X has the form given in Theorem 9.5.10(ii) with $p = 2$ and $S_1 \neq \emptyset$, in which case

$$\mathcal{M} \cap \mathcal{U} = \{\Gamma \setminus S_j\}_{j \geq 2}. \tag{90}$$

For this, we assume that (89) does not hold. Then there must exist $S \in \mathcal{M}$ and $F \in \mathcal{M}_2$ so that $F \cap S \neq \emptyset$ and $F \cap S^c \neq \emptyset$. Otherwise, if for every $F \in \mathcal{M}_2$ one of the two intersections is always empty, then we can show that $S = \cup\{F : F \in \mathcal{M}_2, F \cap S^c = \emptyset\}$ so that $S \in \mathcal{U}$. Note that by Remark 9.5.11, our decomposition for X cannot satisfy case (i) of Theorem 9.5.10. Now let R be a maximal fiber which contains F. Then $R \cap S \neq \emptyset$, and by Proposition 9.5.8, there exists p, $1 \leq p \leq \infty$ such that $X = \ell^p(X_{R\setminus S}, X_{R\cap S}, X_{S\setminus R})$. The sets $F \cap (R \setminus S)$ and $F \cap (R \cap S)$ cannot be empty, so let j, k be elements of those sets, respectively. Then for any scalars s, t, we have

$$\nu(se_j + te_k) = (|s|^p + |t|^p)^{1/p}.$$

However, $j, k \in F \in \mathcal{M}_2$, so we must have $p = 2$. Hence we have our decomposition satisfying case (ii) of Theorem 9.5.10 with

$$X = \ell^2((X_j)_{j\geq 2}).$$

Let F be a maximal ℓ^2 fiber. If $F \cap S_j = S_j$ for some $j \geq 2$, then $X_j = \ell^2(\text{card}(S_j))$ with $\text{card}(S_j) \geq 2$. This contradicts 9.5.10 (ii)(c). Hence let us suppose that F intersects both S_j and S_k with $k, j \geq 2$ and $k \neq j$ and that

$F \cap S_j \neq S_j$. Suppose $r_k \in F \cap S_k$ and let $x \in X_j$. Then

$$\begin{aligned}\nu(s) &= \nu\left(\sum_{i \in F \cap S_j} x(i)e_i + \sum_{i \in F^c \cap S_j} x(i)e_i\right)\\ &= \nu\left(\nu\left(\sum_{i \in F \cap S_j} x(i)e_i\right) e_{r_k} + \sum_{i \in F^c \cap S_j} x(i)e_i\right)\\ &= \left(\nu\left(\sum_{i \in F \cap S_j} x(i)e_i\right)^2 + \nu\left(\sum_{i \in F^c \cap S_j} x(i)e_i\right)^2\right)^{1/2}.\end{aligned}$$

The second equality holds above because F is a fiber, and the third equality holds since $r_k \notin S_j$ and X is the ℓ^2 sum of the X_j's. The result is that $X_j = \ell^2(X_{F\cap S_j}, X_{F^c \cap S_j})$ which again contradicts the fact that X_j cannot be so decomposed.

Therefore we see that F cannot intersect more than one S_j, and since under the circumstances S_1 is an ℓ^2 fiber, each $N \setminus S_j$ is in $\mathcal{U}$ for $j \geq 2$. Hence $\mathcal{M} \neq \{N \setminus S_j\}_{j \geq 2}$ only if $S_1 \neq \emptyset$.

Now we we wish to show that (87) holds.

Case (1). Assume that X has the form described in 9.5.10 (i). Then $\mathcal{M} = \{S_j\}_{j \in \Gamma}$ and by (89), $\mathcal{M} \subset \mathcal{U}$. Since $\widetilde{T}(\mathcal{M}) = \mathcal{M}$ by (88), we see that there must be a permutation π of Γ so that

$$\widetilde{T}(S_{\pi(j)} = S_j.$$

If we let T_j denote the restriction of T to the subspace $X_{\pi(j)}$, we obtain the desired result.

Case (2). Suppose that X has the form given in 9.5.10 (ii). Then as we saw in Remark 9.5.11 (iii),

$$\mathcal{M} = \{N \setminus S_j\}_{j \geq 2} \cup \{N \setminus \{s\}_{s \in S_1}\}.$$

If $S_1 = \emptyset$, then $\mathcal{M} \subset \mathcal{U}$ and the proof is the same as in Case (1). Assume, then, that $S_1 \neq \emptyset$. We must consider two subcases, depending on whether $p = 2$ or not.

Case (2a). If $p \neq 2$, then by (89) and (88), $\widetilde{T}(\mathcal{M}) = \mathcal{M}$. In this case, $\{s\} \in \mathcal{M}_2$ for every $s \in S_1$, and by the isometry theorems of Section 9.2 or 9.3 we conclude that $\widetilde{T}(\{s\})$ has cardinality 1 and is in $\mathcal{M}_2$. Since $N \setminus \{s\}$ is a maximal fiber for each $s \in S_1$, we must have $\widetilde{T}(N \setminus \{s\}) = N \setminus \widetilde{T}(\{s\}) \in \mathcal{M}$. Thus, $\widetilde{T}(\{N \setminus S_j\}_{j \geq 2} = \{N \setminus S_j\}_{j \geq 2}$ and there is a permutation π of Γ so that $\pi(1) = 1$ and $\widetilde{T}(N \setminus S_{\pi(j)}) = S_j$.

Case (2b). If $p = 2$, then $S_1 \in \mathcal{M}_2$ and $\mathcal{M} \cap \mathcal{U} = \{N \setminus S_j\}_{j \geq 2}$ by (90). By (88), $\widetilde{T}(\mathcal{M} \cap \mathcal{U}) = \mathcal{M} \cap \mathcal{U}$, from which we conclude again that there is permutation π of Γ with $\pi(1) = 1$ and $\widetilde{T}(N \setminus S_{\pi(j)}) = N \setminus S_j$ for $j \geq 2$. Since $S_1 \in \mathcal{M}_2$, it is true that $\widetilde{T}(S_1) \in \mathcal{M}_2$. Furthermore, since $S_1 = \cap_{j \geq 2}(N \setminus S_j)$, we have

$$\widetilde{T}(S_1) = \cap_{j\geq 2}\widetilde{T}(N \setminus S_j) = \cap_{j\geq 2}(N \setminus S_j) = S_1.$$

Again it follows that T must have the desired form. □

Let us consider the space $X_{p,q}(4)$, that is, the four-dimensional version of the space $X_{p,q}$ that was defined earlier following Definition 9.5.1. The norm equivalence classes are $\{1, 2\}, \{3\}$, and $\{4\}$. The decomposition given by Theorem 9.5.5 would involve three spaces, $\ell^p(2)$ and two one-dimensional spaces and the space $E = X_{q,p}(3)$. An isometry T on the space would map $\ell^p(2)$ to itself and presumably could interchange the one-dimensional spaces. However $S_1 = \{1, 2, 3\}$ and $S_2 = \{4\}$ form a partition of maximal fibers so by Theorem 9.5.12, the fiber space X_{S_1} is mapped to itself by T and X_{S_4} is mapped to itself, which shows that T must fix both of the one-dimensional spaces in the first decomposition. Hence Theorem 9.5.12 gives us additional information.

As one more example, let $Y = (\mathbb{C}^3, \nu_1)$, where ν_1 is the norm defined by (82). Then if $X = Y(Y)$, there are no symmetries for ν_1 except the identity, and so the only isometries on X are "diagonal."

Our theorem does yield an interesting corollary concerning symmetric spaces. A space is *symmetric* if the symmetry class for the norm includes all permutations. Because an isometry on a direct sum space must map the summands among themselves, any permutation of the original indices which does not preserve the summands cannot be an isometry. Hence, we have the following result.

9.5.13. Corollary. *(Li and Randrianantoanina) If X is a complex symmetric sequence space, then $X(X)$ is symmetric if and only if $X = \ell^p$ for some p, $1 \leq p \leq \infty$.*

As we remarked earlier, the real case presents difficulties so that even in the absence of ℓ^2 fiber spaces, the isometries need not be permutation isometries nor need the direct sum structure be preserved. (See the example given by (86).) Although some things can be said about the real case, we have chosen not to give an exposition here. We will satisfy ourselves with the remark that Theorem 9.5.12 does hold for any real sequence space with maximal fibers whose isometry group is contained in the group of permutation isometries.

9.6. Notes and Remarks

The idea of forming new Banach spaces by taking direct sums of other spaces surely goes back a long way. There are many ways to present the formal definition and even more variations on the notation. We chose to use the substitution space notion given by Day [**104**], and the notation is generally that used by Li and Randrianantoanina [**245**]. Some of the language is borrowed from the paper of Schneider and Turner [**337**], which forms the basic foundation for nearly everything in this chapter. We have considered our spaces to be sequence spaces with an absolute norm, which can be expressed by requiring the coordinate vectors to form a one-unconditional basis. Of course, lying in the background, but never really made explicit, is the fact that any Banach space with a one-unconditional basis may be regarded as a sequence space with an absolute norm.

The notion of one-unconditional or hyperorthogonal basis is crucial to the entire development, and Definition 9.1.1 is a lifting of one of many possible equivalent formulations given by Singer [**347**, p. 558]. The definition as given focuses on the fact that the norm of an element depends only on the absolute values of the coefficients of the basis vectors. Such a basis is necessarily unconditional and the so-called unconditional basis constant

$$C = \sup_{\theta} \|M_\theta\|,$$

where $\theta = \{\theta_\alpha\}_{\alpha\in\Gamma}$ is a collection of ± 1's and M_θ is the operator defined by

$$M_\theta \left(\sum_{\alpha\in\Gamma} a_\alpha x_\alpha \right) = \sum_{\alpha\in\Gamma} a_\alpha \theta_\alpha x_\alpha,$$

is equal to 1 [**259**]. Hence the designation *one-unconditional basis* seems to be more prevalent than *hyperorthogonal*.

Sequence Space Decompositions. This section reflects the work of Schneider and Turner [**337**], whose paper on finite-dimensional decompositions we regard as the foundation of the entire chapter. In fact, we have exposited the work in an infinite-dimensional setting and many of the statements and arguments come from [**127**] and [**128**]. The notion of equivalent coordinates is from Schneider and Turner. In light of the discussions in Section 5, we could have used the language ℓ^2 equivalence, but we have stuck with the original language, at least until Section 5. The proof of Lemma 9.2.2 is from [**337**].

Lemma 9.2.3 and its proof are taken from Theorem 20.2 in [**347**]. We should remark that a number of authors have contributed to results about absolute norms on finite-dimensional spaces, but we will not try to credit them here. The reader can consult the list of references in [**337**] if interested. The remark following the proof of Lemma 9.2.6 is explicitly stated and proved in [**240**].

We have adapted and made heavy use of results in Sections 3 and 4 of [**337**] to prove Lemmas 9.2.7 and 9.2.8. What we have called a *d-admissible* semi-inner product in Definition 9.2.5 was referred to as *sufficiently ℓ^p-like* in [**127**] and elsewhere in the literature. The fact that an admissible sequence space E has a d-admissible s.i.p. if and only if E has no equivalent coordinates (Lemma 9.2.8 and Corollary 9.2.10) has not been stated explicitly before. That it might be true was suggested to us by an unknown referee many years ago. Incidentally, our use of the term *admissible* for a sequence space whose coordinate vectors form a normalized one-unconditional basis is not widely used, although we used the term in [**134**] for a sequence space with the additional property that there were no equivalent coordinates. We have introduced the term *pure* to describe that situation. That word was also used by Rosenthal [**327**] in the context of skew-Hermitian operators, and we make the connection in Section 9.4.

In talking about Hermitian operators, we have referred to Theorem 5.2.6 (from Volume 1), which gives alternate characterizations. These results are based on Lemma 5.2.5, and we wish to take this opportunity to correct a couple of lines in the proof of that lemma which was given in Volume 1. The first three lines of proof we gave for part (ii) of that lemma are incorrect as stated. Given a bounded operator T on a Banach space X with semi-inner product $[\ ,\ ]$, and $x \in X$ with $\|x\| = 1$, we have for $t > 0$,

$$\begin{aligned}\|(I - tT)x\| &\geq |[(I - tT)x, x]| \\ &\geq \Re[(I - tT)x, x] \\ &= 1 - t\Re[Tx, x] \\ &\geq 1 - t\beta,\end{aligned}$$

where $\beta = \sup\{\Re\lambda : \lambda \in \text{numerical range of } T\}$. Hence for any x we have

$$\|(I - tT)x\| \geq (1 - t\beta)\|x\|,$$

which was the result needed. This argument is essentially suggested in [**49**, pp. 17-18].

The fact that $e^{i\theta}\alpha + e^{-i\theta}\beta + \gamma$ is real for all choices of θ implies $\alpha = \overline{\beta}$, which was also used in Chapter 8, is very important to the proof of Theorem 9.2.9 and seems to have first been used in that way by Tam [**365**].

Corollary 9.2.11 is our version of Theorem 6.2 in [**337**] and it is a generalization of Theorem 5.2.13 [**16**], [**365**]. An alternate proof of the theorem in the finite-dimensional case is given by Partington [**296**]. Spaces satisfying the hypotheses of Theorem 9.2.9 were called spaces of class $\mathcal{S}$ in [**126**], [**127**], and [**128**]. Note that the setting of Theorems 9.2.9 and 9.2.12 is more general than beginning with an admissible sequence space. In particular, the Hilbert spaces could be nonseparable in this setting. Berkson and Sourour [**45**] have obtained the conclusion of Theorem 9.2.9 when X is an ℓ^p sum of Banach spaces.

The characterization of isometries on spaces of class $\mathcal{S}$ (Theorem 9.2.12) was given in [**128**]. Theorem 9.2.13 and its notation were suggested by the finite-dimensional version of Schneider and Turner [**337**]. Note that what we called a *permutation isometry* would be of the form

$$Tx(n) = \lambda_n x(\pi(n)), \tag{91}$$

where π is a permutation of the indices which is a symmetry of the norm. These arise when the summand spaces are one-dimensional. The first result along these lines was, of course, given by Banach [**18**, p. 179] for (real) ℓ^p. Tam [**365**] and Arazy [**16**] in the complex case and Braverman and Semenov [**54**] in the real case obtained the result for symmetric sequence spaces which were not equal to ℓ^2 . Rolewicz [**324**, Theorem IX 8.3, 8.5] gives the same result for both real and complex symmetric spaces. The term *elementary* has also been given to permutation isometries on sequence spaces [**310**]. This fits with our use of that term in Chapter 5.

One of the applications of these isometry characterizations for sequence spaces is to describe isometries on Orlicz sequence spaces [**365**], [**128**], [**126**] as we did in Theorem 5.2.14. (See also [**213**].) Crucial to the description is whether the sequence space part has equivalent coordinates, which means that the defining function φ is quadratic on some interval. Indeed, the equivalence classes determined by equivalent coordinates consist of a class N_1 together with a family of singletons. Thus, a reflexive Orlicz sequence space can be written as the sum of an ℓ^2 space and a space with no equivalent coordinates. Let us record this formally.

9.6.1. THEOREM. *Let ℓ^φ be a reflexive Orlicz space on a purely atomic measure space and not equal to a Hilbert space. Then $\ell^\varphi = \ell^2(N_1) + \ell^{\varphi'}$ where $\ell^{\varphi'}$ has no equivalent coordinates, and where N_1 is either empty (in which case this term is absent) or has at least two elements. Every isometry on ℓ^φ is of the form $T = T_1 + T_2$, where T_1 is unitary on ℓ^2 and T_2 is a permutation isometry, that is, satisfies (91).*

Using this we can get a new version of Theorem 5.2.14 which eliminates the assumption about a symmetric basis. Note that it is possible for the sequence space to contain two-dimensional Hilbert subspaces even though there are no equivalent coordinates, but this does not affect the nature of the isometries. (See [**128**] for details.)

Other papers which are related to the material in this section include [**114**], [**176**], and [**238**].

Hermitian Elements and Orthonormal Systems. Nearly everything in this section comes from the oft-referenced paper of Kalton and Wood [**212**], and the theorems we have attributed to them are in that cited paper. The main goal here was to get Theorems 9.3.14 and 9.3.15 [**212**, Theorem 6.1, 6.2], which are analogues of Theorems 9.2.12 and 9.2.9, respectively. The form of a space as a decomposition of Hilbert spaces, as in the latter two theorems, is carried out in a powerful way for general Banach spaces in Theorem 9.3.12. Our Theorem 9.3.13 ties the two sections together.

Orthonormal systems were studied by Berkson [**42**] with a slightly different approach than by Kalton and Wood, although the two definitions are shown to be equivalent in Section 7 of [**212**]. A result from Berkson's paper was used in the proof of Proposition 9.3.2. Berkson obtained a number of results about Hermitian projections. He proved Corollary 9.3.9 (Corollary 4.4 in [**212**] for spaces of dimension greater than 2 [**42**, Theorem 2.22]), although he did not use the term *Hermitian element*.

We included some of the details about splittings and Hermitian decompositions because we thought them very interesting. An example is the useful fact that the closed linear span Y of an orthonormal system is orthogonal and so yields a Hermitian decomposition when Y contains no subspace isomorphic to c_0, which is based on some results of Bessaga and Pelczynski [**46**]. It is striking that in any complex Banach space X there is a unique subspace $\hat{h}(X)$ which is the closed linear span of any maximal orthonormal system, and that it may be decomposed into a direct sum of Hilbert components.

We should remark that in the final two sections of [**212**], Kalton and Wood tackle a question of Rolewicz [**324**] concerning *maximal* norms. A norm is maximal if its group of isometries is maximal. They show, among other things, that the norm on $C[0, 1]$ is maximal. We will have more to say on this subject in Chapter 12.

The Case for Real Scalars: Functional Hilbertian Sums. As we have observed a number of times, the real case usually presents problems. This section is an exposition of a good share of the 1986 paper of Rosenthal [**327**], which establishes analogues for real spaces of the results of the previous sections. The formal study of the Lie algebra of a Banach space, the collection of skew-Hermitian operators, seems to have begun with Rosenthal [**326**], although other authors, including Legisa [**239**], Robbin [**320**], and Vidav [**372**], have related articles.

The development of Hilbert components is available through the idea of *well-embedded* Hilbert subspaces. This notion appeared earlier in a series of papers by Legisa [**235**], [**236**], and [**234**], and was apparently first defined by Vidav [**372**]. We note that if $k \sim j$ (as defined in Section 2), then $sp\{e_k, e_j\}$ is well embedded and, conversely, if the e_j's are from a one-unconditional basis. This figures in the proof of Theorem 9.4.13, which is the only theorem in this section which is not given pretty much directly in [**327**].

The proof of Lemma 9.4.7 is given in [**327**, Lemma 2.8], and this is an important step in the proof of Theorem 9.4.8, the crucial fact that distinct Hilbert components are orthogonal. We mentioned, but did not prove, the statement that if X is a subspace of Z with the property that every one-dimensional subspace of X is orthogonally complemented in Z, then X is a well-embedded Hilbert subspace of Z. This is stated as Theorem 2.2 in [**327**]. The proof in the complex case is implicit in the work of Kalton and Wood [**212**] and therefore in the work in Section 9.3. Berkson [**42**, (1.4)] showed that P is an orthogonal projection if and only if P is Hermitian. The hypotheses therefore imply that every element of X is Hermitian. Then $X \subset h(Z)$ and it must be contained in one of the Hilbert components. Theorem 9.3.13 shows it is well embedded. The proof for the real case is given by Rosenthal, but it involves some considerable effort. When $X = Z$, the result for complex spaces is due originally to Berkson (Corollary 9.3.9 commented on in the previous subsection), and in the real case it is due to Saint-Raymond, according to Rosenthal [**327**, p. 433].

The language of unconditional sum and functional unconditional sum is Rosenthal's language for the decomposition of real spaces, and we have adopted his notation for such sums. Theorem 9.4.9 corresponds to Theorem 3.1 in [**327**], and Corollary 9.4.10 is Theorem 1 in the introduction of Rosenthal's paper. The description of the Lie algebra of an FHS space, Theorem 9.4.14, is a combination of Theorem 3.8 of [**327**] and one of the remarks following it.

Rosenthal points out [**327**, p. 420] that he was partially motivated by the following question,

$$\text{If } Y \text{ is a complemented subspace of a Banach space } X \\ \text{with an unconditional basis, must } Y \text{ have an unconditional basis?} \tag{92}$$

Because of renorming, one could rephrase the problem by assuming that Y is one-complemented, or one could assume that X has a one-unconditional basis. In fact, it can be shown that if Y is a one-complemented subspace of a complex space X with one-unconditional basis, then Y has an unconditional basis. Rosenthal [**325**] has shown that this result is implicit in the work of Kalton and Wood [**212**]. (See also [**138**] and [**313**].) The corresponding statement for real spaces is not true [**39**]. The original question (92), given in [**259**], is still open as far as we know.

Rosenthal's study of real space decompositions that we have featured in this section can often be used to obtain results for the complex case as well. We would like to illustrate this by proving the above-mentioned result, which is given by Rosenthal [**327**, Corollary 3.16].

9.6.2. THEOREM. *(Rosenthal) Let Z be a complex Banach space and suppose $Z = \overline{sp}\{TZ : rank\, T = 1 \text{ and } T \in \mathfrak{A}(Z)\}$. Then the space Z has a one-unconditional basis (and conversely). Furthermore, if X is a one-complemented subspace of Z, then X has a one-unconditional basis.*

PROOF. Let $Y = Z_{\mathbb{R}}$ denote the space Z regarded as a real Banach space. The hypotheses imply that $Y = \overline{sp}\{TY : \text{rank } T = 2 \text{ and } T \in \mathfrak{A}(Y)\}$, so that by Corollary 9.4.10, Y is FHS. We claim that if H is a Hilbert component for Y, then it is a complex linear space and a Hilbert component for Z as well. To see this, consider the operator U defined on $Y = Z_{\mathbb{R}}$ by $Uz = iz$. Then U is skew-Hermitian so that $UH \subset H$ by Theorem 9.4.11, and we must conclude that H is a subspace of Z. Also, U is an isometry and must preserve the orthogonal complement $\mathcal{O}(H)$. Then both H and $\mathcal{O}(H)$ are complex linear, and H is a complex Hilbert space. If a and b are real numbers with $|a + ib| = 1$, then $aI + bU$ is an isometry when restricted to H, so that for $x \in H$ and $y \in \mathcal{O}(H)$, we have

$$\|(aI + bU)x + y\| = \|x + y\|$$

since H is well embedded. We conclude that H is orthogonally complemented by $\mathcal{O}(H)$ in Z and is well embedded. This is enough to establish our claim. Now if we suppose that $\{H_\lambda\}$ are the Hilbert components of Y, they must also be the Hilbert components for Z, and by Theorem 9.3.12, Z has an orthonormal (one-unconditional) basis.

Suppose that X is one-complemented in Z, where now we have that Z has a one-unconditional basis. Then $X_{\mathbb{R}}$ is one-complemented in $Z_{\mathbb{R}}$. It can be shown that there is, for each $x \in X_{\mathbb{R}}$, a skew-Hermitian operator U on $Z_{\mathbb{R}}$ such that $Ux \in X$ and $U^2x = -x$. It follows from [**327**, Theorem 3.15] that $X_{\mathbb{R}}$ is FHS, and by the argument given in the previous paragraph, that $X_{\mathbb{R}}$ being FHS is enough to guarantee that X has a one-unconditional basis.

□

The examples with which we closed the section were given by Rosenthal [**327**] to illustrate some of the earlier results. Again we point out that it is not the presence of two-dimensional Hilbert subspaces that affect the decompositions, but rather the "equivalence" of coordinates.

Decompositions with Banach Space Factors. We return in this section to consideration of complex sequence spaces with absolute, standardized norm. The primary reference for this section is the 1998 paper of Li and Randrianantoanina [**245**], and it is there that the definition of a fiber (Definition 9.5.1) is first given. Proposition 9.5.8 [**245**, Proposition 2.2] is the foundation for the decomposition of a space as sums of Banach spaces. Crucial also is the beautiful theorem of Bohnenblust (9.5.3). In their proof of Proposition 2.2 of [**245**], Li and Randrianantoanina had utilized the theorem of Bohnenblust in its original form (see Remark 9.5.6 (i)). We could not see how one could know that $\nu(se_j + te_k) = \nu(te_j + se_k)$, which was necessary to satisfy the hypotheses of Bohnenblust's theorem. After being questioned about this, Randrianantoanina responded with the inductive argument given in step 1 of the proof of Theorem 9.5.3. This improvement of Bohnenblust's original theorem [**48**, Theorem 4.1] eliminates the need for the troubled hypothesis. Thus, in the proof of Theorem 9.5.3, step 1 is due to Randrianantoanina, and the remainder of the proof is as given originally by Bohnenblust.

The idea of norm equivalence is our small contribution, but it does give an alternate type of decomposition which is possible to obtain, even when there are no maximal fibers. Part of this discussion will appear in [**125**]. The proof of Theorem 9.5.7 that the isometries must permute the "fiber" spaces is based on the same idea as the proof of Theorem 9.5.12, which is due to Li and Randrianantoannna [**245**, Theorem 3.1]. We emphasize again, that although these theorems are generalizations of Theorems 9.2.12 and 9.3.14, their proofs depend on the special results for the Hilbert space decompositions. In [**134**], spaces were considered in the form of Theorem 9.2.12 except that the spaces X_j are not necessarily Hilbert spaces. Theorems characterizing isometries on such spaces were given (Theorems 2.5 and 3.7) which required special conditions on the summands. The conclusions of both of these theorems follow from Theorem 9.5.7 provided the summand spaces have one-unconditional bases, since in this case the big space will have a one-unconditional basis. The special conditions required in [**134**] were needed because the method of proof there utilized the form of the Hermitian operators. In the proof of Theorem 9.2.12, it was important that $A\mathcal{H}(X_n)B = 0$ implies either $A = 0$ or

$B = 0$, where $\mathcal{H}(X_n)$ represents the Hermitian operators on X_n. On a general Banach space, this does not hold.

We want to mention one other example here of the characterization of the isometry group of a (real) space with one-unconditional basis. The space is called Tsirelson's space, and the isometries are shown to be essentially diagonal with positive or negative ones as multipliers, except that the first two coordinates may be permuted. This result is found in the treatment of Tsirelson's space by Casazza and Shura [**85**, p. 29], although they actually attribute the proof to Beauzamy and Casazza. It is interesting that the first two coordinates are norm equivalent, and all the other equivalence classes are singletons. Hence, the characterization in [**85**] fits into the form predicted by Theorem 9.5.7. In addition, the proof shows that the only symmetry of the underlying space is the identity.

When maximal fibers exist, it is possible to obtain the decomposition detailed in Theorem 9.5.10 [**245**, Theorem 2.4]. The space $E_p(2)$ given in Definition 9.5.9 was first defined by Lacey and Wojtaszczyk [**228**]. It is the presence of spaces like this, which are, in the real case, isometric to $\ell^p(2)$, which cause trouble in describing isometries on real spaces, as is shown by the example in (86) and following.

Corollary 3.5 of [**245**], in which isometries of $X(Y)$ are described for symmetric spaces X, Y, does not seem to be quite true as stated, since a space such as $\ell^1(2)(\ell^2(2))$ would have isometries other than those advertised. If Y has an isometry group contained in the group of permutation isometries, then the conclusion would hold. We note that any space with "proper" norm equivalence classes (that is, not a singleton nor the entire index set) can not be symmetric since if, say, N_1 is an equivalence class with at least two elements j, k and N_2 is another class containing m, then the permutation interchanging k and m and fixing the other indices cannot be an isometry. Thus, if X is symmetric and not an ℓ^p space, then the space $Z = X(X)$ must fit into case (i) of Theorem 9.5.10, so that it cannot be symmetric. This yields Corollary 9.5.13 [**245**, Corollary 3.6].

Li and Randrianantoanina conclude their paper with a study of the group of isometries on real spaces of the form $X(Y)$ where X, Y are finite-dimensional symmetric spaces with $\dim Y \geq 2$. As pointed out in the text, Theorem 9.5.12 does hold for any real sequence space with maximal fibers whose isometry group is contained in the group of permutation isometries. It is pointed out in [**245**] that such spaces include

(i) spaces with Δ-bases ([**150**]), and
(ii) spaces which are p-convex with constant 1 for $2 < p < \infty$ ([**310**]).

We end with a list of a few more papers that are related to material in this section: [**91**], [**147**], [**249**], [**309**], [**348**], [**350**], [**351**], [**378**], and [**379**].

CHAPTER 10

Matrix Spaces

10.1. Introduction

The space $M_{m,n}(\mathbb{C})$ of all $m \times n$ complex matrices can be made into a Banach space in a number of ways and so provides a fertile ground for the study of isometries. The earliest norm to be considered on this matrix space was the familiar operator norm given by

$$\|A\| = \sup \frac{\|Ax\|_2}{\|x\|_2},$$

where $x \in \mathbb{C}^n$ and $\|\cdot\|_2$ denotes the Euclidean norm. As early as 1925, the isometries on $M_{m,n}$ were described by Schur, and in the next section we will give Morita's proof of that result. It may be that this theorem of Schur is the very first one that characterizes the isometries of a specific Banach space.

The problem of finding isometries for various norms can be placed within a very general context of preserver problems as follows. Suppose f is a continuous real-valued function defined on k nonnegative variables $t_1, \ldots, t_k$. If $A \in M_{m,n}$ and A^* denotes its conjugate transpose, let the nonnegative square roots of the eigenvalues of the $n \times n$ matrix A^*A (the singular values of A) be denoted by

$$s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A).$$

If we define the function $\hat{f}(A)$ by

$$\hat{f}(A) = f(s_1(A), \ldots s_k(A)), \;\; 1 \leq k \leq n,$$

a very general question asks for a description of all linear transformations T mapping $M_{m,n}$ into itself for which

$$\hat{f}(T(A)) = \hat{f}(A). \qquad (93)$$

For those f which are positive definite, it is not difficult to show that the set of all T which satisfy (93) is a group under multiplication, and that any such T is nonsingular and satisfies

$$\hat{f}(T^{-1}(A)) = \hat{f}(T(T^{-1}(A))) = \hat{f}(A).$$

Note that if we take f above to be given by $f(t) = t$ and $k = 1$, then the corresponding $\hat{f}$ is just the so-called *spectral norm*, which is the same as the operator norm.

In this chapter we want to consider isometries for a whole class of norms, called the (p,k) norms, defined on $M_{m,n}$ as follows: for $1 \leq k \leq n, 1 \leq p \leq \infty$, let

$$\nu_{p,k}(A) = \left(\sum_{j=1}^{k} s_j(A)^p \right)^{1/p}. \tag{94}$$

In the case $p = \infty$, we mean, as usual, the maximum of the k singular values, which by our notation and assumption above is $s_1(A)$. These norms are *unitarily invariant*, which means that for any p, k, and unitary operators U, V in M_m, M_n, respectively, we have

$$\nu_{p,k}(UAV) = \nu_{p,k}(A)$$

for all $A \in M_{m,n}$. By M_k we mean $M_{k,k}$, the square matrices of order k. In the particular case when $p = 1$, the $\nu_{1,k}$ norms are denoted by ν_k or $\|A\|_k$ and called *Ky-Fan norms*. Note that the case $k = 1$ is the spectral or operator norm, and the case of $k = m = n$ is the *trace norm*.

When $m = n$, the isometries T on M_n for the $\nu_{1,1} = \nu_1$ norm, the operator norm, are, by Kadison's theorem (Chapter 6) of the form

$$T(A) = UAV \text{ or } T(A) = UA^tV, \tag{95}$$

where U, V are unitaries and A^t represents the transpose of A. This description will appear throughout this chapter. As indicated, we shall give the proof for $M_{m,n}$ with the operator norm (Schur's theorem) in Section 2. The case for $m = n$ and the trace norm will be left to the next chapter where the infinite-dimensional case will be given. In fact, the cases for $k = m = n$ and arbitrary $1 \leq p \leq \infty, p \neq 2$, the Schatten $\mathcal{C}_p$ spaces, are also covered in Chapter 11. In Section 3 we will establish the theorem of Marcus on operators that preserve the unitary group, and then develop the theorem of Grone and Marcus which gives the isometries for $m = n$ and $p = 1$. In Section 4 we will treat the case where we may have $m \neq n$ and p other than 1. This involves contributions from Grone and also Li and Tsing.

10.2. Morita's Proof of Schur's Theorem

We think it interesting, for historical reasons, to give the proof of Schur's characterization of isometries of $M_{m,n}$ with respect to the operator norm. Morita says that Schur's theorem is an immediate corollary of the following lemma.

10.2.1. Lemma. *(Morita) Suppose T is a linear transformation from the matrix space $M_{m,n}(\mathbb{C})$ to itself for which $T(Z)$ has rank 1 if Z has rank 1, and $T(Z)$ has rank at least 2 if the rank of Z is 2. Then there exist nonsingular matrices A, B, of orders m,n, respectively, such that*

(i) *$T(Z) = AZB$ for all $Z \in M_{m,n}$ if $m \neq n$, and*
(ii) *$T(Z) = AZB$ or $T(Z) = AZ^tB$ for all Z if $m = n$.*

We are not going to prove this lemma here, since a similar result will be proved in the next chapter in connection with the characterization of the isometries on the Schatten $\mathcal{C}_p$ spaces (including the infinite-dimensional cases.)

We are going to prove the theorem below, however, pretty much as Morita gave it. Note that throughout this chapter, I will denote the identity matrix of the appropriate order.

10.2.2. THEOREM. *(Schur, 1925) Suppose T is a linear isometry of the matrix space $M_{m,n}(\mathbb{C})$ endowed with the operator norm. Then there exist unitary matrices U, V of orders m,n, respectively, such that for $m \neq n$, we have $T(Z) = UZV$ for all $Z \in M_{m,n}$. If $m = n$, then we may have $T(Z) = UZV$ or $T(Z) = UZ^tV$ for all $Z \in M_{m,n}$.*

PROOF. We will give the proof under the assumption that $m \geq n$, but the opposite case would be treated similarly. For a given scalar λ and $Z \in M_{m,n}$, let

$$\varphi(\lambda; Z) = det[\lambda I - Z^*Z] \text{ and } \psi(\lambda; Z) = \varphi(\lambda, T(Z)).$$

For $Z = [z_{jk}]$, where $z_{jk} = x_{jk} + iy_{jk}$, we may regard $\varphi(\lambda, Z)$ and $\psi(\lambda, Z)$ as polynomials with coefficients in the ring

$$\mathbb{C}[x_{11}, x_{21}, \ldots, x_{mn}, y_{11}, y_{21}, \ldots, y_{mn}] = \mathbb{C}[x, y],$$

thinking of the x_{jk} and y_{jk} as independent indeterminates. Let $R(\varphi, \psi)$ denote the *resultant* of $\varphi(\lambda; x, y)$ and $\psi(\lambda; x, y)$. Since T is an isometry, $\varphi(\lambda; Z)$ and $\psi(\lambda; Z)$ have at least one root in common for each Z. Hence, $R(\varphi, \psi)$ is the zero element of $\mathbb{C}[x, y]$ and $\varphi(\lambda, x, y)$ and $\psi(\lambda; x, y)$, as elements of $\mathbb{C}[x, y][\lambda]$, must have a common factor. However, the polynomial $\varphi(\lambda; x, y)$ is irreducible. For suppose not. Then there are polys g, h in $\mathbb{C}[x, y][\lambda]$ such that

$$\varphi(\lambda; x, y) = g(\lambda; x, y)h(\lambda; x, y).$$

Choose $x_{jk} = 0$ for $j > n$ and $y_{jk} = 0$ for all j, k. Let X denote the corresponding matrix, so that

$$\varphi(0; X) = (-1)^n(\det[x_{jk}])^2.$$

Hence by the factorization we have assumed, and since $\det[x_{jk}]$ is irreducible in $\mathbb{C}[x_{11}, \ldots, x_{nn}]$, we have either

$$g(0; X) = (-1)^n\omega \det[x_{jk}], \quad h(0; X) = \omega^{-1} \det[x_{jk}]$$

or

$$g(0; X) = (-1)^n\omega, \quad h(0; X) = \omega^{-1}(\det[x_{jk}])^2,$$

where ω is a scalar. However, both of these cases are impossible, and we conclude that $\varphi(\lambda; x, y)$ is irreducible. Since it shares a factor with $\psi(\lambda; x, y)$, the two must really be equal. Thus, the Hermitian matrices Z^*Z and $T(Z)^*T(Z)$ are equivalent (since they have the same eigenvalues) and as a result we may conclude that Z and $T(Z)$ have the same rank. Now by Lemma 10.2.1, there exist nonsingular matrices A, B such that $T(Z) = AZB$ (or possibly $T(Z) = AZ^tB$ if $m = n$). We may (by the singular value decomposition) choose unitary matrices U_1, U_2, V_1, and V_2 of the appropriate order such that $U_1AU_2 = A_1$ and $V_2BV_1 = B_1$ are positive diagonal matrices. Therefore,

$$T(Z) = U_1^{-1}A_1U_2^{-1}ZV_2^{-1}B_1V_1^{-1}$$

for all $Z \in M_{m,n}$. Since $\|UZ\| = \|Z\|$ for any unitary U, we have

$$\begin{aligned}\|Z\| = \|U_2ZV_2\| &= \|T(U_2ZV_2)\| \\ &= \|A_1ZB_1\|\end{aligned}$$

for all Z. It follows from this that A_1 and B_1 must be scalar matrices and the scalar is 1. We conclude that

$$T(Z) = U_1^{-1}U_2^{-1}ZV_2^{-1}V_1^{-1}$$

and the proof is complete. □

10.3. Isometries for (p, k) Norms on Square Matrix Spaces

We separate the characterization of isometries for (p, k) norms on $M_{m,n}$ into five cases:

(i) $k = 1$,
(ii) $k = m = n,\ p = 1$,
(iii) $k = m = n, p \neq 2$.
(iv) $1 \leq k \leq m = n,\ p = 1$, and
(v) $1 \leq k \leq \min\{m, n\}\ p \neq 2$.

The first case is the one we handled in the previous section, and the second and third items will be covered in the next chapter where the setting includes the infinite-dimensional case. In this section we will examine case (iv). We begin with a theorem of Marcus that will be used in the sequel, but is very interesting in its own right.

10.3.1. THEOREM. *(Marcus) Suppose T is a linear transformation on $M_n(\mathbb{C})$ for which $T(U)$ is unitary for each unitary U in M_n. There exist unitary operators U, V such that*

$$T(A) = UAV \ \ or \ \ T(A) = UA^tV \ \ for\ all \ \ A \in M_n.$$

PROOF. We may assume that $T(I) = I$. Suppose that A is a skew-Hermitian operator (i.e., $A^* = -A$) so that for each real t, the matrix

$$T(\exp(tA)) = \sum (t^j/j!)T(A^j)$$

is a unitary matrix. By writing out the fact that $[T(\exp(tA))]^*T(\exp(tA)) = I$, we obtain many relations among $T(A^j)$ and $T(A^j)^*$. In particular, we have

$$T(A) + T(A)^* = 0 \ \text{ and}$$

$$T(A^2) + T(A^2)^* = 2T(A)^2.$$

From the first of these two statements above, we get that $T(A)$ is skew-Hermitian whenever A is skew-Hermitian. Because T is linear, it now follows that $T(A)$ is Hermitian whenever A is Hermitian. Since A^2 must be Hermitian if A is skew-Hermitian, we conclude from the second displayed equality above that $T(A^2) = T(A)^2$. One can easily verify that for any pair A, B of Hermitian matrices, we must have

$$T(AB + BA) = T(A)T(B) + T(B)T(A).$$

From this, and the fact that any matrix $Z \in M_n$ may be written as $Z = A + iB$, where A, B are Hermitian, we see that

$$T(Z^2) = T(Z)^2 \text{ and } T(Z^*) = T(Z)^*.$$

Thus T is what is called a C^*-isomorphism or Jordan*-isomorphism (see Chapter 6). Kadison [**209**] showed that a Jordan*-isomorphism is the direct sum of a *-isomorphism and a *-anti-isomorphism, and in the current setting must be one or the other. It is known (see, for example, [**115**, p. 128]) that any *-isomorphism is of the form $T(A) = UAU^*$ and *-anti-isomorphism is of the form $T(A) = UA^tU^*$. □

We are going to consider (p,k) norms where $p = 1$, so we will have

$$\nu_k(A) = \sum_{j=1}^{k} s_j(A),$$

where $1 \leq k \leq n$ and $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A) \geq 0$ are the singular values of A. We will often just write s_j and suppress the A when the meaning is clear. First, however, we are going to establish a lemma that gives a calculation method for the general (p,k) norms.

10.3.2. LEMMA. *(Grone and Marcus) For $1 \leq p \leq \infty$ and $1 \leq k \leq n$,*

$$\nu_{p,k}(A) = \max_{x_t, U} \left(\sum_{j=1}^{k} |\langle UAx_t, x_t\rangle|^p \right)^{1/p}, \tag{96}$$

where the maximum is over all orthonormal sets $x_1, \ldots, x_k$ in $\mathbb{C}^n$ and unitary $U \in M_n(\mathbb{C})$.

PROOF. By using polar decomposition, we may replace A in (96) by $H = (A^*A)^{1/2} \geq 0$. Then

$$(97)\quad \left(\sum_{t=1}^{k} |\langle UHx_t, x_t|^p\right)^{1/p} = \left(\sum_{t=1}^{k} |\langle H^{1/2}x_t, H^{1/2}U^*x_t\rangle|^p\right)^{1/p}$$
$$\leq \left(\sum_{t=1}^{k} \|H^{1/2}x_t\|_2^p \|H^{1/2}U^*x_t\|_2^p\right)^{1/p}$$
$$\leq \left\{\left(\sum_{t=1}^{k} \|H^{1/2}x_t\|_2^{2p}\right)^{1/2} \left(\sum_{t=1}^{k} \|H^{1/2}U^*x_t\|_2^{2p}\right)^{1/2}\right\}^{1/p}$$
$$\leq \left(\sum_{t=1}^{k} \langle Hx_t, x_t\rangle^p\right)^{1/2p} \left(\sum_{t=1}^{k} \langle HU^*x_t, U^*x_t\rangle^p\right)^{1/2p}.$$

If we select an orthonormal basis of eigenvectors of H which corresponds to $s_1, \ldots, s_n$, extend the original orthonormal set $x_1, \ldots, x_k$ to an orthonormal basis $x_1, \ldots, x_n$, and write out both x_t and Hx_t in terms of the basis of eigenvectors, we see that

$$\langle Hx_t, x_t\rangle = \sum_{j=1}^{n} b_{tj}s_j, \quad t = 1, \ldots, n$$

where $B = [b_{tj}]$ is an n-square doubly stochastic matrix (i.e., both row and column sums are one). By the Minkowski inequality, the function on the right in the equality

$$\left(\sum_{t=1}^{n} \langle Hx_t, x_t\rangle^p\right)^{1/p} = \left\{\sum_{t=1}^{n} \left(\sum_{j=1}^{n} b_{tj}s_j\right)^p\right\}^{1/p}$$

is convex as a function of B. It is a result of Birkhoff that the extreme points for the set of doubly stochastic matrices are the permutation matrices, and it follows that the maximum value of

$$\left(\sum_{t=1}^{k} \langle Hx_t, x_t\rangle^p\right)^{1/p} = \left\{\sum_{t=1}^{k} \left(\sum_{j=1}^{n} b_{tj}s_j\right)^p\right\}^{1/p}$$

is given by

$$\left(\sum_{t=1}^{k} s_j^p\right)^{1/p}.$$

The same argument will establish this inequality for the right-most term in the last line of (97), and by multiplying the two together, we get that the last line in (97) is bounded by

$$\left(\sum_{t=1}^{k} s_j^p\right)^{1/p} = \nu_{p,k}(A).$$

Since $\nu_{(p,k)}(A)$ may be achieved by $(\sum_{t=1}^{k} |\langle UAx_t, x_t\rangle|^p)^{1/p}$, equation (96) is established.

□

It is worth noting that if the rank of A $(=\rho(A))$ is 1, then $\nu_{p,k}(A) = s_1(A)$ is independent of both p and k. Also, if $p = 2, k = n$, then $\nu_{2,n}$ is the Euclidean norm induced by the inner product

$$\langle A, B\rangle = tr(B^*A),$$

which is sometimes called the Frobenius norm or the Hilbert-Schmidt norm.

Our goal now is to show that isometries for the Ky Fan norms must have the canonical form given by (95). We will need some preliminary results which will identify the extreme points of the unit ball B_k in M_n for the k norm, ν_k. We note that all of these results are due to Grone and Marcus.

10.3.3. Lemma. *Suppose that $1 \le k < n$, $\Delta = [\Delta_{jk}]$ is an upper triangular matrix with zero main diagonal, and D is a diagonal matrix. Then*

(i) *$\nu_k(\alpha I + \Delta) \ge k|\alpha|$ with equality if and only if $\Delta = 0$;*
(ii) *if $\nu_k(D) = \nu_k(2I - D) = k$, then $D = I$.*

Proof. (i) The result is trivial if $\alpha = 0$ and we may assume $\alpha = 1$. As usual, e_j will denote the unit vector with 1 in the jth position and zeros elsewhere. From Lemma 10.3.2, we have

$$\nu_k(I + \Delta) \ge \sum_{t=1}^{k} |\langle (I + \Delta)e_t, e_t\rangle| = k.$$

If for some $j \ne k$, we have $\Delta_{jk} \ne 0$, let $x_1 = (e_j + e_k)/\sqrt{2}$, and let $x_2, \ldots, x_k$ be chosen from among $e_1, \ldots, e_n$, excluding e_j and e_k. Let U be a unitary diagonal matrix such that the entry from the jth row and kth column is $|\Delta_{jk}|$. If we apply Lemma 10.3.2 again, we obtain

$$\begin{aligned}\nu_k(I + \Delta) &= \nu_k(I + U^*\Delta U)\\ &\ge \sum_{t=1}^{k} |\langle (I + U^*\Delta U)x_t, x_t\rangle|\\ &= |1 + \langle U^*\Delta U x_1, x_1\rangle| + k - 1\\ &= k + \frac{|\Delta_{jk}|}{2}.\end{aligned}$$

This establishes (i).

(ii) Write $D = \operatorname{diag}(a_1, \ldots, a_n)$, where we assume that $|a_1| \geq \cdots \geq |a_n|$. Suppose $|a_1| > |a_n|$. Since $\nu_k(D) = |a_1| + \cdots + |a_k|$ is the largest sum of k of the $|a_j|$, we must have

$$|a_{n-k+1}| + \cdots + |a_n| < k.$$

From this inequality and Lemma 10.3.2 one more time, we see that

$$\begin{aligned}\nu_k(2I - D) &\geq \left| \sum_{t=n-k+1}^{n} \langle (2I - D)e_t, e_t \rangle \right| \\ &= |2k - (a_{n-k+1} + \cdots + \alpha_n)| \\ &\geq 2k - (|a_{n-k+1} + \cdots + |a_n|) \\ &> k.\end{aligned}$$

Thus, if $\nu_k(2I - D) = k$, we conclude that

$$|a_1| = \cdots = |a_n| = 1. \tag{98}$$

In the same way, we can argue that the main diagonal entries of $2I - D$ have the same modulus $(D = 2I - (2I - D))$. Hence,

$$|2 - a_1| = \cdots = |2 - a_n| = 1. \tag{99}$$

Putting (98) and (99) together, we obtain $D = I$. □

10.3.4. Lemma.

(i) *If $1 \leq k < n$ and U is unitary, then $k^{-1}U$ is an extreme point of B_k.*
(ii) *If $k = n$, $n^{-1}U$ is not an extreme point of B_n.*
(iii) *For any $k, 1 \leq k \leq n$, if A is an extreme point of B_k, then either A has rank 1 with $\nu_k(A) = 1$, or $A = k^{-1}U$ where U is unitary.*
(iv) *If A has rank 1 and $\nu_k(A) = 1$, then A is an extreme point of B_k if and only if $k > 1$.*

Proof. (i) Obviously, $\nu_k(UA) = \nu_k(A)$ for any unitary U, so it suffices to show this for $U = I$. Suppose, then, that $I = (A + B)/2$, where $\nu_k(A) \leq k$ and $\nu_k(B) \leq k$. In fact, we must have both $\nu_k(A) = k$ and $\nu_k(B) = k$ since $\nu_k(I) = k$. Since $B = 2I - A$, it is the case that A, B commute and so may be simultaneously brought to upper triangular form with a unitary similarity. Therefore, we may assume

$$A = D + \Delta$$

and

$$B = (2I - D) - \Delta,$$

where D is diagonal and Δ is strictly upper triangular. Now

$$k = \nu_k(A) = \nu_k(D + \Delta) \geq \nu_k(D) \tag{100}$$

and

$$k = \nu_k(B) = \nu_k((2I - D) - \Delta) \geq \nu_k(2I - D). \tag{101}$$

From (100), (101), and the triangle inequality, we have

$$2k = \nu_k(2I) \leq \nu_k(D) + \nu_k(2I - D) \leq 2k.$$

It follows that both (100) and (101) are equalities so that by Lemma 10.3.3 (ii), $D = I$, and (100) becomes

$$\nu_k(I + \Delta) = k.$$

Therefore, by Lemma 10.3.3 (i), $\Delta = 0$, from which we conclude that $A = B = I$, and $k^{-1}U$ is an extreme point.

(ii) For this we again assume $U = I$, and write

$$n^{-1}I = n^{-1}(I - E_{nn}) + n^{-1}E_{nn},$$

where E_{jk} is the matrix whose (j, k) entry is 1 and the remaining entries are 0. If we let $A = n^{-1}(I - E_{nn})$ and $B = n^{-1}E_{nn}$, we can show that $\nu_k(A) + \nu_k(B) = 1$, and since

$$n^{-1}I = \nu_n(A)\frac{A}{\nu_n(A)} + \nu_n(B)\frac{B}{\nu_n(B)},$$

we see that $n^{-1}I$ is not extreme.

(iii) Suppose $A \in B_k$ is neither rank 1 nor a multiple of a unitary matrix. Because of the invariance of ν_k under unitary equivalence, we may assume $A = \text{diag}(s_1, \ldots, s_n)$, where $s_1 \geq \cdots \geq s_n$ are the singular values of A. The conditions on A imply that

$$s_1 + \cdots + s_n = 1,$$
$$\text{rank}(A) = m > 1,$$
$$s_1 > s_n.$$

The plan is to express $A = C + D$, where C and D are linearly independent and $\nu_k(C) + \nu_k(D) = 1$. This will show that

$$A = \nu_k(C)\frac{C}{\nu_k(C)} + \nu_k(D)\frac{D}{\nu_k(D)},$$

showing that A is not extreme. Let $E_m = E_{11} + \cdots + E_{mm}$. We consider three cases.

Case 1. ($s_1 > s_m$.) Put $C = s_m E_m$ and $D = A - C$. Since $s_1 > s_m$, the matrices C, D are linearly independent. It is straightforward to show that $\nu_k(C) = \min\{m, k\}s_m$, and that $\nu_k(D) = 1 - \nu_k(C)$.

Case 2. ($s_1 = s_m, m \geq k$.) Since $s_1 > s_n$, we must have $m < n$. Let $C = \frac{1}{2}(A + s_m E_{nn})$ and $D = \frac{1}{2}(A - s_m E_{nn}) = A - C$. Then $\nu_k(C) = \frac{1}{2} = \nu_k(D)$.

Case 3. ($s_1 = s_m, m < k$.) This time let $C = s_1 E_{11}$ and $D = A - C$. Then $\nu_k(C) = s_1 = 1/m$ and $\nu_k(D) = (m-1)s_1 = (m-1)/m$.

(iv) Suppose A has rank 1 and $\nu_k(A) = 1$. Once again, by the unitary invariance of ν_k, we may assume that $A = E_{11}$. If $k = 1$, we let $C = \frac{1}{2}(E_{11} + E_{22})$ and $D = \frac{1}{2}(E_{11} - E_{22})$, we have $\nu_1(C) = \frac{1}{2} = \nu_1(D)$, and A is clearly not an extreme point.

On the other hand, if $k > 1$, we suppose $A = E_{11} = \frac{1}{2}(C + D)$, where $\nu_k(C) = \nu_k(D) = \nu_k(E_{11}) = 1$. From this we infer that

$$\begin{aligned} 1 &= \nu_1(E_{11}) \\ &\leq \frac{1}{2}\nu_1(C) + \frac{1}{2}\nu_1(D) \\ &\leq \frac{1}{2}\nu_k(C) + \frac{1}{2}\nu_k(D) \\ &= 1. \end{aligned}$$

Note that the inequality $\nu_1(C) \leq \nu_k(C)$ is strict unless the rank of C is 1. Hence, the above calculation implies that both C and D have rank 1. In this case, the $\nu_{2,n}$ norm is the same as the ν_k norm for E_{11}, C, and D, from which we have

$$\begin{aligned} 1 &= \nu_{2,n}(E_{11}) \\ &\leq \frac{1}{2}\nu_{2,n}(C) + \frac{1}{2}\nu_{2,n}(D) \\ &\leq \frac{1}{2}\nu_k(C) + \frac{1}{2}\nu_k(D) \\ &= 1. \end{aligned}$$

The $\nu_{2,n}$ norm is Euclidean, and the triangle inequality above is strict unless $C = rD$ for some $r > 0$. Since $\nu_k(C) = \nu_k(rC) = 1$, we have $r = 1$ and $C = D$, which, at last, finishes the proof of the lemma. □

Let us summarize what we have done in the following theorem. For ease in writing, let $\mathcal{R}_k$ denote the matrices A with rank 1 and for which $\nu_k(A) = 1$, and let $\mathcal{U}$ denote the set of unitaries.

10.3.5. THEOREM. *(Grone and Marcus) The extreme points of the unit ball B_k for the norm ν_k in M_n are:*

(i) $\mathcal{U}$ *if* $k = 1$;
(ii) $\mathcal{R}_n$ *if* $k = n$;
(iii) $\mathcal{R}_k \cup k^{-1}\mathcal{U}$ *if* $1 < k < n$.

The advertised theorem about isometries is now within reach.

10.3.6. THEOREM. *(Grone and Marcus) If $T : M_n\mathbb{C}) \to M_n(\mathbb{C})$ is a linear transformation and $1 \leq k \leq n$ is a fixed integer for which*

$$\nu_k(T(A)) = \nu_k(A)$$

for all $A \in M_n$, then there exist unitary operators U, V such that

$$T(A) = UAV \quad \text{or} \quad T(A) = UA^tV \quad \text{for all} \quad A \in M_n. \tag{102}$$

PROOF. If $k = 1$, then T must map extreme points to extreme points, and by part (i) of Theorem 10.3.5, T must map unitaries to unitaries. By Theorem 10.3.1, T must have the form in (102). (Note that this is a proof of Schur's theorem.)

If $k = n$, we have the trace norm, and the result will be seen again in Chapter 11. However, we cannot resist giving a proof here, since we have the machinery available and the argument can also be used for the next part. Note that by Theorem 10.3.5 (ii), T must map rank 1 matrices to rank 1 matrices. By Morita's theorem, Theorem 10.2.1, $T(A)$ must have the form PAQ or PA^tQ, where P, Q are nonsingular. Suppose $P = U_1D_1V_1$ and $Q = U_2D_2V_2$ are the singular value decompositions, where the U's and V's are unitary and $D_1 = \text{diag}(\alpha_1, \cdots, \alpha_n)$, $D_2 = \text{diag}(\beta_1, \ldots, \beta_n)$ are diagonal matrices with the respective singular values distributed in descending order on the diagonals. It is straightforward, using the fact that T is an isometry for ν_n and the unitary invariance, to show that

$$\nu_n(A) = \nu_n(D_1AD_2).$$

If $A = E_{11}$, we get

$$1 = \nu_n(E_{11}) = \nu_n(\alpha_1\beta_1E_{11}) = \alpha_1\beta_1.$$

Similarly, we get $1 = \alpha_n\beta_n$, and because of the ordering of these positive scalars, we must conclude that $\alpha_1 = \cdots = \alpha_n = \alpha$, and $\beta_1 = \cdots = \beta_n = 1/\alpha$. Hence we have $P = \alpha U$, $Q = (1/\alpha)V$, where U, V are unitaries. Therefore, $T(A) = UAV$. (A similar argument works in the alternate case.)

Finally, suppose $1 < k < n$. Our goal here is to show that T must map rank 1 matrices to rank 1 matrices so that the above argument can be applied to obtain (102). Suppose that $A \in \mathbb{R}_k$ but $T(A) = cU$ for some unitary U. This is the only other possibility by Theorem 10.3.5 (iii), and the fact that an isometry must map extreme points to extreme points. As before, we may assume $A = E_{11}$. If s, t are scalars, not both zero, then $\rho(sE_{11} + tE_{12}) = 1$. Therefore,

$$T(sE_{11} + tE_{12}) = scU + tT(E_{12}) \tag{103}$$

is either rank 1, or a multiple of a unitary matrix. We observe that U and $T(E_{12})$ are linearly independent. Suppose that $T(E_{12}) = B$, with $\rho(B) = 1$. If we multiply (103) by U^*, neither its rank nor the fact that it is a multiple of a unitary matrix is affected. Thus the matrix $scI + tU^*B$ either has rank 1 or is a multiple of a unitary matrix for all s, t not both zero. But this cannot be true, so we conclude that $T(E_{12}) = B$ must be unitary. However, $scI + tU^*B$ cannot be a multiple of a unitary since U and B are linearly independent. Neither can it have rank one for all s, t not both zero. These contradictions mean that $\rho(A) = 1$ implies that $\rho(T(A)) = 1$, and concludes the proof.

□

10.4. Isometries for (p, k) Norms on Rectangular Matrix Spaces

In this section we consider transformations on $M_{m,n}$ where $m < n$ and so we will be handling case (v) of the list given at the beginning of the previous section. It is perhaps useful to note that if $A \in M_{m,n}$, and has rank k, then A may be written in the form $A = VCW^*$, where $V \in M_m, W \in M_n$ are unitary and $C = [c_{jt}]$ with $c_{11} \geq \cdots c_{kk} > c_{k+1,k+1} = \cdots = c_{mm} = 0$. The c_{jj}'s are the singular values of A.

Although the first order of business is to describe the isometries for the Ky Fan norms where $p = 1$, we digress to obtain some interesting results for $p > 1$. The first involves an alternate way to write the (p, k) norms.

For $p \geq 1, 1 \leq k \leq m$, let

$$\phi_{p,k}(A) = \max_{U,V} \|d_k(UAV)\|_p, \tag{104}$$

where $d_k(A) = (a_{11}, a_{22}, \ldots a_{kk})$, $\|\cdot\|_p$ denotes the norm on $\ell^p(k)$, and the max is taken over all pairs of unitaries of the appropriate sizes. Lemma 10.3.2 shows that

$$\phi_{p,k}(A) = \nu_{p,k}(A)$$

when $m = n$. In the general case it is easy to see from the definition that we always have

$$\nu_{p,k}(A) \leq \phi_{p,k}(A).$$

The fact that equality holds is a theorem due to Fan (and also appears in the book of Gohberg and Krein).

10.4.1. THEOREM. *(Fan) (Gohberg and Krein) If $A \in M_{m,n}$ for $m \leq n$,*

$$\phi_{p,k}(A) = \nu_{p,k}(A)$$

for $p \geq 1, 1 \leq k \leq m$.

It is also clear that if $\alpha_1, \ldots \alpha_k$ are any k diagonal elements of A that

$$\left(\sum_{j=1}^{k} |\alpha_j|^p \right)^{1/p} \leq \phi_{p,k}(A). \tag{105}$$

It is of interest to examine when equality holds in (105). First we need a special lemma.

10.4.2. LEMMA. *(Grone) If*

$$\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix},$$

$p > 1, 0 \leq a \leq 1$, U, V are unitary and

$$\|d_2(UAV)\|_p = \nu_{p,k}(A),$$

then

(i) *UV is diagonal if $a = 1$;*
(ii) *either both of U, V are diagonal or both of U, V have zero diagonal, if $a < 1$.*

PROOF. When $a = 1$ we have $\|d_2(UV)\|_p = 2^{1/p}$, and using this, and the fact that U, V are unitary, one can show that UV is diagonal. When $a = 0$, $\|d_2(UAV)\|_p = 1$, and calculation shows that U, V are both diagonal or both have zero diagonal. Let us suppose that $0 < a < 1$. Let $x = Ve_1, y = Ve_2$ denote the columns of V and suppose $W = VU$. Then

$$\begin{aligned}
\|d_2(UAV)\|_p &= [|\langle UAVe_1, e_1\rangle|^p + |\langle UAVe_2, e_2\rangle|^p]^{1/p} \\
&= [|\langle Ax, W^*x\rangle|^p + |\langle Ay, W^*y\rangle|^p]^{1/p} \\
&= \left[|\langle A^{1/2}x, A^{1/2}W^*x\rangle|^p + |\langle A^{1/2}y, A^{1/2}W^*y\rangle|^p\right]^{1/p}.
\end{aligned} \tag{106}$$

Now by the Cauchy-Schwartz inequality, we have

$$\begin{aligned}
|\langle A^{1/2}x, A^{1/2}W^*x\rangle|^p &\le \langle A^{1/2}x, A^{1/2}x\rangle^{p/2}\langle A^{1/2}W^*x, A^{1/2}W^*x\rangle^{p/2} \\
&= \langle V^*AVe_1, e_1\rangle^{p/2}\langle UAU^*e_1, e_1\rangle^{p/2}
\end{aligned}$$

and

$$\begin{aligned}
|\langle A^{1/2}y, A^{1/2}W^*y\rangle|^p &\le \langle A^{1/2}y, A^{1/2}y\rangle^{p/2}\langle A^{1/2}W^*y, A^{1/2}W^*y\rangle^{p/2} \\
&= \langle V^*AVe_2, e_2\rangle^{p/2}\langle UAU^*e_2, e_2\rangle^{p/2}.
\end{aligned}$$

We apply the Cauchy-Schwartz inequality again to the right-hand sides in each of the above displays to get that their sum is less than or equal to the product of the two expressions

$$[\langle V^*AVe_1, e_1\rangle^p + \langle V^*AVe_2, e_2\rangle^p]^{1/2}$$

and

$$[\langle UAU^*e_1, e_1\rangle^p + \langle UAU^*e_2, e_2\rangle^p]^{1/2}.$$

If we put this together with (106), we get

$$\begin{aligned}
\|d_2(UAV)\|_p &\le \|d_2(V^*AV)\|_p^{1/2}\|d_2(UAU^*)\|_p^{1/2} \\
&\le \nu_{p,2}(A) = \|d_2(UAV)\|_p.
\end{aligned}$$

The fact that equality holds everywhere above can be shown to hold if and only if $U = DV^*$, where D is diagonal. In fact we may assume that $U = V^*$, and so

$$\begin{aligned}
\|d_2(UAV)\|_p &= [\langle Ax, x\rangle^p + \langle Ay, y\rangle^p]^{1/p} \\
&= \left[(|x_1|^2 + a|x_2|^2)^p + (|y_1|^2 + |y_2|^2)^p\right]^{1/p}.
\end{aligned}$$

This last expression represents a strictly convex function of the doubly stochastic matrix

$$\begin{bmatrix} |x_1|^2 & |y_1|^2 \\ |x_2|^2 & |y_2|^2 \end{bmatrix}.$$

Again by the Birkhoff result mentioned in the previous section, asserting that the extreme points are the permutation matrices, we conclude that the maximum occurs only when

$$V = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$$

is diagonal or has a zero diagonal, and $U = DV^*$ must have the same property. □

As a consequence of this lemma we have the following.

10.4.3. LEMMA. *(Grone) If $A \in M_{m,n}(\mathbb{C})$, $p > 1$ and*

$$\|d_n(A)\|_p = \nu_{p,n}(A),$$

then A is diagonal.

PROOF. If A is not diagonal, we may assume (without loss of generality) that $a_{12} \neq 0$. Let

$$A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

By the singular value decomposition, there exist 2×2 unitary matrices U_2, V_2 and nonnegative real numbers $\alpha \geq \beta$ such that

$$U_2 A_2 V_2 = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}.$$

By Lemma 10.4.2, we must conclude that

$$\alpha^p + \beta^p = (\nu_{p,2})^p > |a_{11}|^p + |a_{22}|^p.$$

If we pre- and postmultiply A by $U_2 \oplus I$, $V_2 \oplus I$ (where I is the identity of order $n-2$) to obtain A', we have

$$\nu_{p,n}(A) \geq \|d_n(A')\|_p > \|d_n(A)\|_p,$$

which establishes the lemma. □

This last lemma is used to prove the following theorem.

10.4.4. THEOREM. *(Grone) If $p > 1$, then A is an extreme point of the unit ball $B_{p,k}$ if and only if $s_k(A) = s_m(A)$.*

We are going to omit the somewhat lengthy proof of this theorem.

Let us turn now to the case where $p = 1$. We are also assuming that $m < n$ in this section. We will follow the same line of attack as we did in the previous section and describe the extreme points of the unit ball $B_{1,k} = B_k$. Since unitary operators are not available in the nonsquare situation, we need a more general notion. Thus we let $\mathcal{U}_{m,n}$ denote those elements U of $M_{m,n}$ for which $UU^* = I$. Also, as before, we will let $\mathcal{R}_k$ denote those matrices $A \in M_{m,n}$ for which $\rho(A) = 1$ and $\nu_k(A) = 1$. We will need the following result, which is analogous to Marcus' Theorem 10.3.1, and whose proof we also omit.

10.4.5. THEOREM. *(Grone) Suppose $m < n$ and T is a linear transformation on $M_{m,n}(\mathbb{C})$ which satisfies $T(\mathcal{U}_{m,n}) \subset \mathcal{U}_{m,n}$. Then there exist fixed unitaries U, V (of appropriate size) such that*

$$T(A) = UAV \text{ for all } A \in M_{m,n}.$$

The next lemma is analogous to Lemma 10.3.4 of the previous section.

10.4.6. LEMMA. *(Grone)*

(i) *If $U \in \mathcal{U}_{m,n}$, then $k^{-1}U$ is an extreme point of B_k if and only if $k < m$.*
(ii) *If $A \in \mathcal{R}_k$, then A is an extreme point of B_k if and only if $k > 1$.*
(iii) *If $A \notin \mathcal{R}_k \cup k^{-1}\mathcal{U}_{m,n}$, then A is not an extreme point of B_k.*

PROOF. (i) We may assume $U = I_m = E_{11} + \cdots + E_{mm}$. Since

$$m^{-1}I_m = m^{-1}E_{11} + m^{-1}(I - E_{mm})$$

we see that $m^{-1}I_m = m^{-1}U$ cannot be an extreme point of B_m.

Assume that $k < m$ and

$$I_m = A + (I - A), \text{ with } \nu_k(A) = \nu_k(I - A) = \frac{k}{2}.$$

Now using this and looking at all possible k-subsets of diagonal elements of A and $I - A$, we conclude that

$$d_m(A) = \frac{1}{2}d_m(I).$$

If

$$A = \left[\begin{array}{c|c} A_1 & A_2 \end{array}\right]$$

with A_1 square, we may replace A with

$$A' = \left[\begin{array}{c|c} V^*A_1V & V^*A_2 \end{array}\right]$$

for any $m \times m$ unitary V. If we apply the previous argument to A', we have

$$d_m(A') = d_m(V^*A_1V) = (\frac{1}{2}, \ldots, \frac{1}{2}), \text{ all } V.$$

As a result we conclude that $A_1 = \frac{1}{2}I \in M_m$. Hence,

$$A = \left[\begin{array}{c|c} \frac{1}{2}I & A_2 \end{array}\right],$$

and it can be shown (using techniques used in the (omitted) proof of Theorem 10.4.4) that $A_2 = 0$. Hence we have shown that $2A = 2(I_m - A) = I_m$, and the conclusion is that $m^{-1}I_m$ is extreme.

(ii) For this argument, we will assume that for $A \in \mathcal{R}_k$, $A = E_{11}$. Observe that for any real number $t, 0 < t < 1$,

$$\nu_1(E_{11} \pm tE_{22}) = 1,$$

so

$$E_{11} = \frac{1}{2}(E_{11} + tE_{22}) + \frac{1}{2}(E_{11} - tE_{22})$$

and E_{11} cannot be extreme.

Suppose, then, that $k > 1$, and that we write

$$E_{11} = A + B$$

where $\nu_k(A) = \nu_k(B) = 1/2$. It is easy to show that

$$s_1(A) = \nu_k(A) = \nu_k(B) = s_1(B) = \frac{1}{2},$$

from which it follows that $\rho(A) = \rho(B) = \rho(A+B) = 1$. Thus,

$$\begin{aligned}\nu_{2,k}(A+B) &= \nu_k(A+B)\\ &= \nu_k(A) + \nu_k(B)\\ &= \nu_{2,k}(A) + \nu_{2,k}(B),\end{aligned}$$

and we must infer that A, B are linearly dependent. Hence, $A = B = \frac{1}{2}E_{11}$.

(iii) The proof of this part goes just the same as the proof of part (iii) of Lemma 10.3.4. □

10.4.7. THEOREM. *(Grone) The extreme points of the unit ball B_k for the norm ν_k in $M_{m,n}$ are:*

(i) $\mathcal{U}_{m,n}$ *if* $k = 1$;
(ii) $\mathcal{R}_k \cup k^{-1}\mathcal{U}_{m,n}$ *if* $1 < k < m$;
(iii) R_k *if* $k = m$.

10.4.8. THEOREM. *(Grone) If $T : M_{m,n}(\mathbb{C}) \to M_{m,n}(\mathbb{C})$ is a linear transformation and $1 \le k \le m < n$ is a fixed integer for which*

$$\nu_k(T(A)) = \nu_k(A)$$

for all $A \in M_{m,n}$, then there exist $m\times m$ and $n\times n$ unitaries U, V, respectively, such that

$$T(A) = UAV \ \ \textit{for all} \ \ A \in M_{m,n}.$$

The proof of this theorem is essentially the same as for the proof of Theorem 10.3.5.

The task we set for ourselves in this chapter was to characterize the isometries for all the (p, k) norms on $M_{m,n}(\mathbb{C})$, and it remains to carry this out in the case that $p > 1$ and $m < n$. There is no mystery about what the theorem should say. In order to obtain it, we will need a couple of lemmas. By $S_{p,k}$ we will denote the surface of the unit ball for the $\nu_{p,k}$ norm.

10.4.9. LEMMA. *(Li and Tsing) Suppose $p > 1$, $1 < k < m$. An element A in $S_{p,k}$ has rank greater than $k-1$ if and only if there exists $B \in S_{p,k}$ such that $B \ne A$, and*

$$rA + (1-r)B \in S_{p,k} \ \ \textit{for any} \ \ 0 \le r \le 1. \tag{107}$$

PROOF. Suppose $A \in S_{p,k}$. Let $A = UDV$ be the singular value decomposition of A, where the diagonal matrix can be written as

$$D = \sum_{j=1}^{m} s_j(A)E_{jj},$$

and where we assume that $s_1(A) \geq s_2(A) \geq \cdots \geq s_m(A)$. If $\rho(A) \geq k$, then $s_k(A) > 0$. Let $B = UD'V$ where

$$D' = \sum_{j=1}^{k} s_j(A)E_{jj} + bE_{k+1,k+1}$$

and b is any number in $[0, s_k(A)]$ distinct from $s_{k+1}(A)$. Then $B \neq A$ and (107) is satisfied.

Next, we suppose $\rho(A) \leq k - 1$. Suppose $B \in S_{p,k}$ with singular values $s_1(B) \geq \cdots \geq s_m(B)$. If $C = (A + B)/2$ we have (by [**146**, p. 48]) that

$$\sum_{j=1}^{l} s_j(C) \leq \frac{1}{2}\sum_{j=1}^{l}(s_j(A) + s_j(B)) \text{ for } l = 1, \ldots, m.$$

It therefore follows from [**146**, p. 37] that

$$(108) \qquad \left(\sum_{j=1}^{k} s_j(A)^p\right)^{1/p} \leq \frac{1}{2}\left(\sum_{j=1}^{k}(s_j(A) + s_j(B))^p\right)^{1/p}$$

$$(109) \qquad \leq \frac{1}{2}\left[\left(\sum_{j=1}^{k} s_j(A)^p\right)^{1/p} + \left(\sum_{j=1}^{k} s_j(B)^p\right)^{1/p}\right]$$

$$= 1.$$

If $C \in S_{p,k}$, then the left- and right-hand sides in the above display are both 1, so that both (108) and (109) are equalities. Again from [**146**, p. 37] we must have

$$s_j(A) = s_j(B) = s_j(C) \text{ for } j = 1, \ldots, k.$$

Since $s_k(A) = 0$, the above actually holds for $j = 1, \ldots, m$. Suppose W, Z are $m \times m$ and $n \times n$ unitaries, respectively, such that

$$\sum_{j=1}^{m} s_j(C)E_{jj} = WCZ = \frac{1}{2}(WAZ + WBZ).$$

If $WAZ = [a_{jl}], WBZ = [b_{jl}]$, then because $|a_{jl}| \leq s_1(A)$, we have $a_{11} = s_1(A) = s_1(C)$, and similarly $b_{11} = s_1(B)$. Furthermore, s_1 is the only entry in the first row and column of WAZ (as is $s_1(B)$ for WBZ.) This can be extended, inductively, to obtain $a_{jj} = s_j(A)$ and $b_{jj} = s_j(B)$ for all j. It follows that $A = B = C$, contrary to the fact that $B \neq A$. Thus we cannot have $C \in S_{p,k}$, and in this case, then, we cannot have (107) satisfied. □

10.4.10. LEMMA. *(Li and Tsing) Suppose $p > 1$ and $p \neq 2$. Then $A \in M_{m,n}$ has rank greater than 1 if and only if there exist nonzero matrices B and C in $M_{m,n}$ such that $A = B + C$ and*

$$(110) \qquad 2[\nu_{p,m}(B)^p + \nu_{p,m}(C)^p] = \nu_{p,m}(B + C)^p + \nu_{p,m}(B - C)^p.$$

PROOF. Let $A = UDV$, where U, V are unitaries of the appropriate size and $D = \sum_{j=1}^{m} s_j(A)E_{jj}$. If $\rho(A) > 1$, then $s_2(A) > 0$. Let $B = s_1(A)UE_{11}V$ and $C = A - B$. Thus B and C are nonzero matrices which satisfy (110), and the "only if" statement is proved.

For the "if" part, we suppose $A = B + C$ as in the statement of the lemma. By arguments due to McCarthy, the condition (110) holds if and only if $BB^*CC^* = 0$ and $B^*BC^*C = 0$. This implies that the matrices B and C (and also B^* and C^*) have disjoint ranges. We can find unitaries W, Z such that

$$WBZ = \sum_{j=1}^{k} s_j(B)E_{jj} \text{ and } WCZ = \sum_{j=k+1}^{m} s_j(C)E_{jj}$$

for some k with $1 \leq k < m$. Hence,

$$\rho(A) = \rho(B) + \rho(C) > 1.$$

□

Here is the theorem.

10.4.11. THEOREM. *(Li and Tsing) Suppose* $p > 1$, $1 < k \leq m \leq n$, *and* $(p, k) \neq (2, m)$. *If* T *is a linear transformation on* $M_{m,n}(\mathbb{C})$ *such that* $\nu_{p,k}(T(A)) = \nu_{p,k}(A)$ *for all* $A \in M_{m,n}$, *then there exist unitaries* U, V *of the appropriate size such that*

$$T(A) = UAV \text{ or, in case } m = n,\ T(A) = UA^tV. \tag{111}$$

PROOF. Let T be an isometry for the $\nu_{p,k}$ norm. First we suppose $p > 1$ and $1 < k < m$. Let C have rank greater than $k-1$ and suppose $A = \frac{1}{\nu_{p,k}(C)}C$. By Lemma 10.4.9, there exists $B \in S_{p,k}$ such that (107) holds. Now $T(A)$ and $T(B)$ are distinct elements of $S_{p,k}$ such that

$$rT(A) + (1 - r)T(B) \in S_{p,k} \text{ for all } 0 \leq r \leq 1.$$

Applying Lemma 10.4.9 again, we have that $T(C)$ has rank greater than $k-1$, so that T must map the set of matrices of rank greater than $k - 1$ into itself. By a theorem of Beasley [**21**, Theorem 4], we may conclude that $T(A)$ must have the form PAQ or (if $m = n$) the form PA^tQ, where P, Q are nonsingular. An argument like that given in the middle part of the proof of Theorem 10.3.6 will establish (111).

Finally, suppose $k = m, p > 1$, and $p \neq 2$. Suppose $\rho(A) > 1$, so that by Lemma 10.4.10, there exist nonzero B, C such that $A = B + C$ and (110) holds. It follows that $T(A) = T(B) + T(C)$, where $B' = T(B)$ and $C' = T(C)$ are nonzero matrices which satisfy

$$2[\nu_{p,k}(B')^p + \nu_{p,k}(C')^p] = \nu_{p,k}(B' + C')^p + \nu_{p,k}(B' - C')^p.$$

We conclude from Lemma 10.4.10 again that $T(A)$ has rank greater than 1. Another application of Beasley's theorem gives the desired result. □

10.5. Notes and Remarks

The theorem of Schur appears in [**338**] and the proof which we give in Section 2 is that of Morita [**285**]. Our introduction of the somewhat general preserver problem comes from the 1970 paper of Marcus and Gordon [**268**], in which they obtain the description of T under fairly general conditions, and which we describe in a later section.

The norms $\nu_{p,k}$ are given by Grone and Marcus [**163**] for $m = n$ and by Grone [**162**] for the more general case, and they remark that the fact that the norms are unitarily invariant comes from work of von Neumann [**374**]. We have changed their notation slightly. The book of Horn and Johnson [**180**] is a good place to look for discussion of matrix norms and related matters.

The name Ky Fan norms for ν_k comes from the fact that Ky Fan [**119**] showed that for matrices A, B (or compact operators in a symmetric norm ideal), $\nu(A) \leq \nu(B)$ for all unitarily invariant norms ν if and only if $\nu_k(A) \leq \nu_k(B)$ for all k.

Morita's Proof of Schur's Theorem. As we mentioned above, the proof given in this section comes from the 1941 paper of Morita [**285**]. It is remarkable that Schur [**338**] would have considered such a question so early, and as we mentioned in the introduction, the result may be the very first one in which the goal was to describe the isometries on a given Banach space. Of course, the formal notion of a Banach space was in its infancy at that time.

The proof we have given comes directly from Morita's paper, which also includes the full proof of Lemma 10.2.1. The use of the concept of the *resultant* of polynomials may be less familiar to today's reader than it was in 1941. A discussion may be found, among other places, in Chapter 4 of van der Waerden's book on modern algebra [**369**].

Isometries for (p, k) Norms on Square Matrix Spaces. Almost everything in this section is based on the 1977 paper of Grone and Marcus [**163**]. The theorem of Marcus (10.3.1) first appeared in [**267**], but the proof as we have given it was supplied by Rais [**306**], who also considered an infinite-dimensional version of Marcus' theorem and isometries with respect to the operator norm. Cheung and Li [**92**] have a version of Marcus' theorem for maps from M_n into M_m. The (p, k) norms as we have given them appeared, perhaps first, in [**163**], although they have been studied in various forms earlier by von Neumann [**374**], von Neumann and Schatten [**336**], and Ky Fan [**119**].

The characterization of extreme points on the unit ball for the Ky Fan norms is very interesting in itself, as are the proofs. The useful Lemma 10.3.2 as well as Lemmas 10.3.3, 10.3.4 and Theorems 10.3.5, 10.3.6 are taken from [**163**]. The roots of Lemma 10.3.2 lie in the paper of Ky Fan [**119**] and before that in [**374**]. It is also discussed in Gohberg and Krein [**146**], and specifically in Marcus and Moyls [**269**]. The theorem of Birkhoff on extreme points for the set of doubly stochastic matrices may be found in [**180**, p. 527].

The rank-preserving theorem of Morita, used in the proof of part (ii) of Theorem 10.3.6, was also proved by Marcus and Moyls [**270**]. Similar invertigations were conducted by Beasley [**20**], who considered operators which preserve a given rank. A more recent paper by Li, Rodman, and Semrl [**246**] concerns rank-preserving transformations from matrices of order $m \times n$ to matrices of order $p \times q$, and shows that the results can be somewhat different.

Cheung, Li, and Poon [**93**] have obtained interesting results for isometries between square matrix spaces of different orders, where the spaces are given the operator norm. Let us state the main result from that paper.

10.5.1. THEOREM. *(Cheung, Li, Poon) Suppose $m \leq 2n-1$, and T is a linear map from $M_n(\mathbb{C})$ to $M_m(\mathbb{C})$ such that $\nu_1(T(A)) = \nu_1(A)$ for all $A \in M_n$. Then $m \geq n$, and there exist unitary elements U, V in M_m and a contractive linear map $f : M_n \to M_{m-n}$ such that T has the form*

$$T(A) = U[A \oplus f(A)]V \quad \text{or} \quad T(A) = U[A^t \oplus f(A)]V.$$

Moreover, if $m \geq 2n \geq 4$, then there exists a norm-preserving linear map $\psi : M_n \to M_m$ that is not of the form displayed above.

The authors of [**93**] remark that the above theorem can be obtained from a theorem describing the linear maps which preserve the norms of the *essentially Hermitian* matrices.

The case $k = n$ in Theorem 10.3.6 was proved also by Russo [**332**].

Isometries for (p,k) Norms on Rectangular Matrix Spaces. The material in this section up through Theorem 10.4.8 comes pretty much directly from Grone's paper [**162**]. Theorem 10.4.5, which describes those operators preserving partial isometries, is taken from [**161**].

The singular value decomposition for nonsquare matrices mentioned in the beginning of this section is discussed in Horn and Johnson [**180**] and a proof is also given by Marcus and Gordon [**268**].

The results for $p > 1$, including Lemmas 10.4.9 and 10.4.10 and Theorem 10.4.11, are taken from the paper of Li and Tsing [**250**]. The results for $k = m = n$ and $p \geq 1$ are the finite-dimensional cases of the Schatten p-classes which will be discussed in the next chapter. The reference to McCarthy in the proof of Lemma 10.4.10 is to his generalization of Clarkson's inequality in [**280**], which will appear again in the next chapter. The Beasley reference is to a later paper [**21**] which generalizes his work mentioned earlier to transformations preserving a range of ranks, rather than just one rank.

The results we have described all were obtained under the assumption that the scalar field was the complex numbers. The real case was considered in the paper of Johnson, Laffey, and Li [**205**]. They showed that a linear isometry for the Ky Fan norm on $M_n(\mathbb{R})$ either has the standard canonical form as we have already seen or, in the case where $n = 4$ and $k = 2$, may be a composition of the standard form with a mapping of the form

$$A \to (A + B_1AC_1 + B_2AC_2 + B_3Ac_3)/2$$

or

$$A \to (DA + B_1DAC_1 + B_2DAC_2 + B_3DAC_3)/2,$$

where D, B_j, C_j are special matrices. There is a good discussion of this plus a very nice survey of results, including some for nonlinear transformations, in the recent paper by Chan, Li, and Sze [**89**], which we heartily recommend. Li, Poon, and Sze [**244**] have characterized isometries for Ky Fan norms where the linear maps are between rectangular matrix spaces of different sizes. As one might guess, the descriptions are a bit more complicated, and we will not state them here.

The paper of Marcus and Gordon [**268**] which laid out the rather general preserver problem discussed in the introduction includes a characterization (in the canonical form of (95)) for transformations preserving the function $\hat{f}$ described there when the underlying function f is concave, symmetric, and strictly increasing in each of its variables. Li and Tsing [**251**] characterized the isometries on the space of all $n \times n$ Hermitian matrices with respect to what they called *unitary similarity invariant* norms. A later paper [**252**] is related to that.

In a very interesting paper by Li, Semrl, and Sourour [**248**], the linear isometries for Ky Fan norms on the space of block triangular matrices are characterized. Let us state the main theorem from that paper.

10.5.2. THEOREM. *(Li, Semrl, and Sourour) Let $\mathcal{A} = \mathcal{T}(n_1, \ldots, n_t)$ and $\mathcal{B} = \mathcal{T}(m_1, \ldots, m_s)$ be block upper triangular algebras in M_n and M_m respectively, and let $\phi : \mathcal{A} \to \mathcal{B}$ be a surjective linear isometry for the Ky Fan k norm ν_k, $(1 < k < n)$. Then $m = n, s = t$, and there exist unitary matrices U and V in $\mathcal{B}$ such that one of the following holds:*

(i) *$n_j = m_j$ for every j, i.e. $\mathcal{B} = \mathcal{A}$, and ϕ has the form $A \to UAV$, $A \in \mathcal{A}$;*

(ii) *$n_j = m_{1+t-j}$ for every j, i.e. $\mathcal{B} = \mathcal{A}^+$, and ϕ has the form $A \to UA^+V$, $A \in \mathcal{A}$.*

Conversely, every map of the form described above is a linear isometry with respect to any unitarily invariant norm.

Note that A^t is replaced in (ii) by A^+, the transpose with respect to the "anti-diagonal."

We conclude with a listing of a few more papers that are closely related to the subject matter in this chapter. Of particular note is the nice survey paper of Li [**242**] on norms on finite-dimensional spaces. Others of interest include [**99**], [**221**], [**233**], and [**247**].

CHAPTER 11

Isometries of Norm Ideals of Operators

11.1. Introduction

Let $\mathcal{H}$ be a separable infinite-dimensional complex Hilbert space and $\mathcal{L}(\mathcal{H})$ the Banach space of bounded linear operators on $\mathcal{H}$ with operator norm. As usual, we denote the inner product on $\mathcal{H}$ by $\langle \cdot, \cdot \rangle$. Since $\mathcal{L}(\mathcal{H})$ is also a Banach algebra we can investigate the structure of the isometries of subalgebras or ideals of $\mathcal{L}(\mathcal{H})$. The isometries of $\mathcal{L}(\mathcal{H})$ are well known to be the product of a unitary and a Jordan*-isomorphism as given by Kadison's Theorem 6.1.1. The question arises as to whether or not the isometries of subalgebras have the same form as isometries of the full algebra. We have seen that this is not the case for the commutative version of $\mathcal{L}(\mathcal{H})$, namely $C(K)$. We considered the form of isometries on certain types of subalgebras of C^* algebras in Chapter 6. In this chapter we consider a more special case of this problem, namely the isometries of various minimal norm ideals of $\mathcal{L}(\mathcal{H})$.

11.1.1. Definition. *The pair $(\mathcal{J}, \nu)$ is said to be a symmetric norm ideal in $\mathcal{L}(\mathcal{H})$ if $\mathcal{J}$ is a two-sided ideal in $\mathcal{L}(\mathcal{H})$ and ν is a norm satisfying the following conditions:*

(i) $\nu(A) = \|A\|$ *for all rank 1 operators,*
(ii) $\nu(UAV) = \nu(A)$ *for every $A \in \mathcal{J}$ and every pair of unitary operators $U, V \in \mathcal{L}(\mathcal{H})$.*

If in addition, we have

(iii) $\mathcal{J}$ *= the norm closure of the finite rank operators,*

then $(\mathcal{J}, \nu)$ is said to be a minimal norm ideal.

By the usual norm symbol $\| \cdot \|$ above we mean the operator norm.

The Schatten class $\mathcal{C}_p$ is the most widely known ideal of operators on a complex Hilbert space and has been the subject of much work. We recall that the trace class is the space $\mathcal{C}_1$ which consists of all compact operators T on $\mathcal{H}$ for which the sum $\sum_{n=1}^{\infty} \langle Te_i, e_i \rangle$ is finite for every orthonormal basis $\{e_i\}_{i=1}^{\infty}$ of $\mathcal{H}$ and its value is the same for every orthonormal basis of $\mathcal{H}$. This common value is called the trace of T and denoted by $tr(T)$. For $1 \leq p < \infty$, $T \in \mathcal{L}(\mathcal{H})$ belongs to $\mathcal{C}_p$ if and only if $(T^*T)^{p/2}$ belongs to $\mathcal{C}_1$. The norm is given by

$$\|T\|_p = [tr((T^*T)^{p/2})]^{1/p}.$$

We take $\mathcal{C}_\infty$ to be the Banach space of all compact operators on $\mathcal{H}$ with the usual operator norm. The dual of $\mathcal{C}_p$, $1 < p \leq \infty$, is isometric to $\mathcal{C}_q$ where $1/p + 1/q = 1$, and the duality is given by

$$\langle\langle T, S\rangle\rangle = tr(TS); \;\; T \in \mathcal{C}_p, \;\; S \in \mathcal{C}_q.$$

The dual of $\mathcal{C}_1$ is, of course, $\mathcal{L}(\mathcal{H})$.

For any pair of unitaries U and V it is easy to see that $\|UTV\|_p = \|T\|_p$ for every $T \in \mathcal{C}_p$. Thus, $\| \; \|_p$ is a unitarily invariant norm.

We now briefly describe the more general symmetric norm ideal. Let $\mathcal{S}$ denote the set of complex sequences with only finite many nonzero terms. A *symmetric gauge function* is a norm ν defined on $\mathcal{S}$ such that:

(i) $\nu(1, 0, 0, 0, ...) = 1$.

(ii) $\nu(\xi_1, \xi_2, ...) = \nu(|\xi_1|, |\xi_2|, ...)$.

(iii) $\nu(\xi_1, \xi_2, ..., \xi_n, 0, 0, ...) = \nu(\xi_{\pi(1)}, \xi_{\pi(2)}, ..., \xi_{\pi(n)}, 0, 0, ...)$
for complex numbers $\xi_1, \xi_2, ..., \xi_n$ and permutations π of the set $\{1, 2, 3, ...,$

Given a sequence, $\xi = \{\xi_n\}$, let $\xi^{(n)} = (\xi_1, \xi_2, ..., \xi_n, 0, 0, \dots)$ and set $\nu(\xi) = \lim_{n\to\infty} \nu(\xi^{(n)})$. Let $\mathcal{S}_\nu = \{\xi \in c_0 : \nu(\xi) < \infty\}$.

Given a compact operator T, the sequence of so-called *s-numbers* of T (or the *singular values*) is the nonincreasing sequence $\{s_i(T)\}$ of eigenvalues of the compact Hermitian operator $|T| = (T^*T)^{1/2}$. The eigenvalues are repeated in this sequence according to their multiplicity. Define $\mathcal{C}_\nu$ to be the collection of all compact operators T on $\mathcal{H}$ with the property that the sequence $\{s_i(T)\}$ belongs to $\mathcal{S}_\nu$. The linear space $\mathcal{C}_\nu$ is an ideal in $\mathcal{L}(\mathcal{H})$ and can be endowed with a symmetric norm by defining $\|T\| = \nu(\{s_i(T)\})$. The space $(\mathcal{J}_\nu, \nu)$ is defined to be the closure in $\mathcal{C}_\nu$ of the finite rank operators. It is well known that $(\mathcal{J}_\nu, \nu)$ is a minimal norm ideal in $\mathcal{L}(\mathcal{H})$ and that each minimal norm ideal $(\mathcal{J}, \nu)$ determines a symmetric gauge function on $\mathcal{S}$.

In Section 2 we characterize the isometries on the Schatten classes $\mathcal{C}_p$, a result first obtained by Arazy. We will exhibit a shorter proof due to Erdos. In Section 3, we treat Sourour's characterization of isometries on minimal symmetric norm ideals. Section 4 will be devoted to a brief study of isometries on the noncommutative L^p spaces, following the work of Yeadon and others. As usual, we close the chapter with some notes and remarks.

Obviously, there are connections between the subject of this chapter and that of Chapter 10. In case $\mathcal{H}$ is finite-dimensional, then our space $\mathcal{L}(\mathcal{H})$ is the matrix space $\mathcal{M}_n$ of the previous chapter, and the norms under consideration include the norms ν_1, ν_n, and $\nu_{p,n}$ which were considered there. We will not use that notation in this chapter.

11.2. Isometries of $\mathcal{C}_p$

One of the first results for the isometries of the Schatten class was obtained by Russo, who used a duality argument along with results of Kadison to establish the form of the surjective isometries of the "trace class," $\mathcal{C}_1$. We give this result for historical completeness. The arguments used in this proof are clever but do not appear to extend to the other ideals.

To state Russo's theorem we will give some additional terminology. Let $\mathcal{C}$ be a C^*-algebra. A linear map α on $\mathcal{C}$ is a *-automorphism if it is an automorphism and has the additional property that $\alpha(T^*) = (\alpha(T))^*$ for every $T \in \mathcal{C}$. The map α is called a *-anti-automorphism if it is a linear, *-preserving, product-reversing map from $\mathcal{C}$ onto itself.

The fundamental result of Kadison on isometries of C^*- algebras which we mentioned in the introduction plays a central role in the proof of Russo's theorem. Although it is treated extensively in Chapter 6, we include the statement again below for easy reference. In the statement of the theorem, a C^*-isomorphism between C^*-algebras $\mathcal{A}$, $\mathcal{B}$ is a linear isomorphism of $\mathcal{A}$ onto $\mathcal{B}$ which preserves self-adjoint elements, powers, and hence "the full Jordan structure" of A. We also refer to it as a Jordan*-isomorphism. In this chapter we will use uppercase Greek letters to represent isometries on the operator spaces.

11.2.1. THEOREM. *(Kadison) A linear isometry Φ of one C^*-algebra $\mathcal{A}$ onto another C^*-algebra $\mathcal{B}$ is a C^*-isomorphism followed by left multiplication by a fixed unitary.*

As we mentioned in the previous chapter, Kadison also showed that a Jordan*-isomorphism was necessarily a direct sum of a *-isomorphism and a *-anti-isomorphism. In fact, in case the corresponding C^*-algebra is a *factor*, that is, has trivial center, then the Jordan*-isomorphism is either a *-isomorphism or a *-anti-isomorphism.

Note: We have used the superscript * on an operator to represent both the Hilbert space adjoint and the Banach space adjoint. We trust the reader will not find that disconcerting in the arguments below.

11.2.2. THEOREM. *(Russo) If Φ is a linear isometry of the Banach space $\mathcal{C}_1$ onto itself, then there exists a *-automorphism or a *-anti-automorphism α of $\mathcal{L}(\mathcal{H})$ and a unitary operator U in $\mathcal{L}(\mathcal{H})$ such that for every $T \in \mathcal{C}_1$,*

$$\Phi(T) = \alpha(TU).$$

PROOF. If Φ is a surjective linear isometry of $\mathcal{C}_1$ then the adjoint Φ^* is an isometry of the dual space $\mathcal{L}(\mathcal{H})$. By the theorem of Kadison 11.2.1, $\Phi^*(A) = U\beta(A)$ where U is unitary and β is a Jordan*-isomorphism, which is, in fact, by our remarks prior to the statement of the theorem, either a *-isomorphism or anti-isomorphism. If Ψ is a linear map on $\mathcal{C}_1$ for which $\Psi^* = \beta$, then it may be shown that $\Phi(T) = \Psi(TU)$. In showing this it seems helpful to note that

$$\langle\langle T, B\rangle\rangle = \langle\langle V^*T, BV\rangle\rangle = \langle\langle TV, V^*B\rangle\rangle$$

for every $T \in \mathcal{C}_1, B \in \mathcal{L}(\mathcal{H})$, and unitary V. This is easy to see when we recall that the duality pairing is given by $\langle\langle T, B\rangle\rangle = tr(TB)$. Also we use the fact that unitaries are preserved by *-isomorphisms and anti-isormorphisms.

To complete the proof it is enough to show that β is the adjoint of β^{-1} restricted to $\mathcal{C}_1$, for then we may take $\alpha = \beta^{-1}$ to obtain the statement in the theorem. Now, as we have seen before, any *-isomorphism β on $\mathcal{L}(\mathcal{H})$ is of the form

$$\beta(A) = VAV^*,$$

where V is unitary. (See, for example, [**115**, p. 128].) Any *-anti-isomorphism will then be of the form VA^tV^*, where A^t denotes the transpose of A. The transpose A^t is defined relative to a fixed basis $\{e_i\}_{i=1}^{\infty}$ by the statement $\langle A^t e_j, e_i\rangle = \langle Ae_i, e_j\rangle$. Hence it is sufficient to consider

(i) $\beta(A) = VAV^*$ with V a fixed unitary operator, or
(ii) with a change of basis, $\beta(A)$ is the transpose of the operator A.

In the first case, letting $\langle\langle A, B\rangle\rangle = tr(AB)$ denote the dual pairing, we see

$$\langle\langle T, \beta(A)\rangle\rangle = \langle\langle T, VAV^*\rangle\rangle = \langle\langle V^*TV, A\rangle\rangle = \langle\langle \beta^{-1}(T), A\rangle\rangle.$$

In the latter case,

$$\langle\langle T, \beta(A)\rangle\rangle = tr(T\beta(A)) = tr(\beta(T)A) = \langle\langle \beta^{-1}(T), A\rangle\rangle.$$

In either case we see that β is the Banach space adjoint of β^{-1} and this completes the proof. □

We note here that by the remark concerning the form of *-isomorphisms and anti-isomorphisms on $\mathcal{L}(\mathcal{H})$ it follows that we could write the conclusion of Russo's theorem in the form:

(i) $\Phi(T) = UTV$ for some unitary operators U, V in $\mathcal{L}(\mathcal{H})$
or
(ii) $\Phi(T) = UT^tV$ for some unitary operators U, V in $\mathcal{L}(\mathcal{H})$.

J. Arazy was the first to obtain a characterization of the surjective isometries of $\mathcal{C}_p$ for $p \neq 1$.

11.2.3. Theorem. *(Arazy) Let Φ be an isometry of $\mathcal{C}_p$ onto itself* $(0 < p \leq \infty)$. *Then either*

(i) $\Phi(T) = UTV$ *for some unitary operators U and V in $\mathcal{L}(\mathcal{H})$*
or
(ii) $\Phi(T) = UT^tV$ *for some unitary operators U and V in $\mathcal{L}(\mathcal{H})$.*

Three proofs of this theorem exist in the literature and in this section we will give the proof due to Erdos. In addition to giving a shorter and more elementary proof than that of Arazy, Erdos showed that the result can be stated without resorting to a transpose relative to some distinguished basis. In order to avoid transposes Erdos needed to introduce the notion of *anti-unitary* transformation on $\mathcal{H}$.

11.2.4. Definition. *Let $\mathcal{H}$ be a complex Hilbert space. A transformation U from $\mathcal{H}$ onto $\mathcal{H}$ is* anti-unitary *if*

(i) $U(x+y) = U(x) + U(y)$ *for every pair $x, y \in \mathcal{H}$,*

(ii) $U(\lambda x) = \bar{\lambda}U(x)$ *for every complex number* λ *and* $x \in \mathcal{H}$,
(iii) $\|U(x)\| = \|x\|$ *for every* $x \in \mathcal{H}$.

These transformations are not linear but with each *anti-unitary* operator an adjoint can be associated. To see that, we suppose that U is an *anti-unitary* operator acting on a complex Hilbert space $\mathcal{H}$. If h is a fixed vector in $\mathcal{H}$, then we can define a bounded linear functional ϕ_h on $\mathcal{H}$ as follows: $\phi_h(k) = \langle h, U(k)\rangle$. The Riesz representation theorem implies that there exists a unique $\tilde{U}(h)$ such that

$$\langle h, U(k)\rangle = \langle k, \tilde{U}(h)\rangle.$$

It is easy to show that
(1) the map $U \to \tilde{U}$ is norm preserving, and
(2) $\tilde{U}(x + \lambda y) = \tilde{U}(x) + \bar{\lambda}\tilde{U}(y)$ for all scalars λ and vectors $x, y \in \mathcal{H}$.

We will call $\tilde{U}$ the adjoint of U even though the relation between U and $\tilde{U}$ is quite different than the usual relationship between an operator and its adjoint. The arguments above show that the adjoint of an *anti-unitary* operator is again *anti-unitary*.

We now state Erdos' version of Arazy's theorem.

11.2.5. Theorem. *(Arazy, Erdos) A map Φ is a surjective linear isometry of $\mathcal{C}_p$ onto itself if and only if there exists unitary operators U_1 and U_2 and anti-unitary operators V_1 and V_2 such that*
(i) $\Phi(T) = U_1 T U_2$
or
(ii) $\Phi(T) = V_1 T^* V_2$
for all $T \in \mathcal{C}_p$.

Before giving the proof of the theorem we will first reconcile the differences in the statements of the two theorems. Arazy's statement of the theorem requires the use of a transpose operation. Let $Cx = \sum_i \langle e_i, x\rangle e_i$. Then $C^2 = I$ and C is *anti-unitary*. A simple computation with finite rank operators shows that $CT^*C = T^t$ and equivalently $T^t = CT^*C$. If U and V are unitary, $UT^tV = (UC)T^*(CV)$, where UC and CV are easily seen to be *anti-unitary*. Therefore Arazy's version of the theorem is the same as Erdos' version.

Both Arazy and Erdos prove that a surjective isometry must preserve the rank of the finite rank operators. The key tool used by both is the following version of Clarkson's inequalities for the norm in $\mathcal{C}_p$ which is due to McCarthy. In the statement below, p' denotes the exponent conjugate to p in the sense that $1/p + 1/p' = 1$. We will omit the proof of this theorem.

11.2.6. Theorem. *(McCarthy) Let S and T be bounded operators on $\mathcal{H}$. Then*
(i) $2^{p-1}(\|S\|_p^p + \|T\|_p^p) \le \|T + S\|_p^p + \|T - S\|_p^p \le 2(\|T\|_p^p + \|S\|_p^p)$ $(0 < p \le 2)$
(ii) $\|T + S\|_p^{p'} + \|T - S\|_p^{p'} \le 2(\|T\|_p^p + \|S\|_p^p)^{p'/p}$ $(1 < p \le 2)$

(iii) $2(\|T\|_p^p + \|S\|_p^p) \leq \|T+S\|_p^p + \|T-S\|_p^p \leq 2^{p-1}(\|T\|_p^p + +\|S\|_p^p)$ $(2 \leq p < \infty)$
(iv) $2(\|T\|_p^p + \|S\|_p^p)^{p'/p} \leq \|T+S\|_p^{p'} + \|T-S\|_p^{p'}$ $(2 \leq p < \infty)$

If $p = 2$, equality always holds; if $p \neq 2$ and the quantities above are finite, equality holds in (i) or (iii) if and only if $TS^* = 0$ and $T^*S = 0$; in (ii) or (iv) if and only if $T = S$ or $T = 0$ or $S = 0$.

In what follows, $\mathcal{F}$ will denote the finite rank operators on $\mathcal{H}$ and $\mathcal{R}$ will denote the rank 1 operators. We first make the elementary observation that $f_1 \otimes g_1 + f_2 \otimes g_2 \in \mathcal{R}$ if and only if the f_i's or the g_i's are linearly dependent. Recall that $f \otimes g$ represents the rank 1 operator defined by $(f \otimes g)(y) = \langle y, f\rangle g$.

11.2.7. Lemma. *Let Φ be a linear map from $\mathcal{F}$ onto $\mathcal{F}$ which preserves rank and is isometric on $\mathcal{R}$ (with respect to operator norm). Then there must exist unitary operators U_1, U_2 and anti-unitary operators V_1, V_2 on $\mathcal{H}$ such that*

(i) $\Phi(S) = U_1 S U_2$ *for all finite rank operators $S \in \mathcal{F}$ or*
(ii) $\Phi(S) = V_1 S^* V_2$ *for all finite rank operators $S \in \mathcal{F}$.*

Proof. Let $e \in \mathcal{H}$ with $\|e\| = 1$. Then $\Phi(e \otimes e) = g \otimes f$ for some unit vectors f and g in $\mathcal{H}$. Let $x \in \mathcal{H}$ be an arbitrary vector. Then $(g \otimes f) + \Phi(x \otimes e) = \Phi((e + x) \otimes e) \in \mathcal{R}$. From this it follows that there exist vectors ξ_x and $\eta_x \in \mathcal{H}$ such that

$$\Phi(x \otimes e) = \begin{cases} \xi_x \otimes f & (I) \\ \text{or} & \\ g \otimes \eta_x & (II). \end{cases}$$

It must be true that either (I) or (II) must hold simultaneously for all vectors, for if we assume that (I) is false for z and (II) is false for x, we have that both $\{\xi_x, g\}$ and $\{f, \eta_z\}$ are linearly independent pairs which cannot hold since $\xi_x \otimes f + g \otimes \eta_z = \Phi(x \otimes e + z \otimes e)$ has rank 1. We now suppose that (I) holds and consider the properties of the map $x \longrightarrow \xi_x$. It follows easily from the above observation that this map is linear. If we define a transformation $U : \mathcal{H} \longrightarrow \mathcal{H}$ by $Ux = \xi_x$, then

$$\|x\| = \|x \otimes e\| = \|\Phi(x \otimes e)\| = \|Ux \otimes f\| = \|Ux\|.$$

To show that U is a unitary operator we must show that U is onto. To that end we suppose that $h \in \mathcal{H}$. Since Φ maps $\mathcal{R} \to \mathcal{R}$ we know that there exists $y, z \in \mathcal{H}$ such that $\Phi(z \otimes y) = h \otimes f$. Now choose $x \in \mathcal{H}$ so that x and z are linearly independent. Since $(Ux + h) \otimes f = \Phi(x \otimes e + z \otimes y)$ it follows that $x \otimes e + z \otimes y \in \mathcal{R}$ and hence $y = \lambda e$ for some scalar λ. The equalities $h \otimes f = \Phi(z \otimes y) = \Phi(z \otimes \lambda e) = \Phi(\bar{\lambda} z \otimes e) = U(\bar{\lambda} z) \otimes f$ imply that $U(\bar{\lambda} z) = h$ and therefore U is onto. Hence, if (I) holds there exists a unitary operator U on $\mathcal{H}$ such that

$$\Phi(x \otimes e) = Ux \otimes f \quad \text{for every} \quad x \in \mathcal{H}. \tag{112}$$

If (II) holds, then by defining $Vx = \eta_x$ and using essentially the same arguments we are led to the existence of an anti-unitary operator V such that

$$\Phi(x \otimes e) = g \otimes Vx \text{ for all } x \in \mathcal{H}. \tag{113}$$

If we consider $e \otimes y$ with variable y, then $g \otimes f + \Phi(e \otimes y) = \Phi(e \otimes e) + \Phi(e \otimes y) = \Phi(e \otimes (e + y))$ implies that $g \otimes f + \Phi(e \otimes y) \in \mathcal{R}$. Therefore, as in the earlier part of the argument, there must exist vectors γ_y and σ_y such that

$$\Phi(e \otimes y) = \left\{ \begin{array}{ll} g \otimes \gamma_y & (I^{'}) \\ \text{or} & \\ \sigma_y \otimes f & (II^{'}) \end{array} \right\},$$

and one or the other must hold for all y. If we suppose that $(I^{'})$ holds, then it can be shown that there exists a *unitary* operator W such that

$$\Phi(e \otimes y) = g \otimes Wy \text{ for all } y \in \mathcal{H}. \tag{114}$$

Similarly if $(II^{'})$ holds, there is an *anti-unitary* operator Q such that

$$\Phi(e \otimes y) = Qy \otimes f \text{ for all } y \in \mathcal{H}. \tag{115}$$

It is clear that (112) and (115) can not hold simultaneously for this would imply that $\Phi(x \otimes e + e \otimes y) = Ux \otimes f + Qy \otimes f \in \mathcal{R}$ and hence $x \otimes e + e \otimes y \in \mathcal{R}$ for all $x, y \in \mathcal{H}$. This is impossible. Likewise, (113) and (114) cannot hold simultaneously. Therefore let us suppose that (112) and (114) hold. Then for any $w, z \in \mathcal{H}$ which are not scalar multiples of e there exist $u, v \in \mathcal{H}$ such that $\Phi(z \otimes w) = u \otimes v$. Now, $\Phi(e \otimes e + z \otimes w) \notin \mathcal{R}$ and $\Phi(z \otimes e + z \otimes w) \in \mathcal{R}$. This implies that $\Phi(e \otimes e) + u \otimes v \notin \mathcal{R}$ and $Uz \otimes f + u \otimes v \in \mathcal{R}$ so that u must be a scalar multiple of Uz. Furthermore, it follows that v is a scalar multiple of Ww.

We are thus lead to the conclusion that

$$\Phi(z \otimes w) = \mu(w, z)[Uz \otimes Ww]$$

for some scalar-valued function $\mu(w, z)$. The fact that $\Phi(z \otimes w)$ is linear in w and conjugate linear in z implies that $\mu(w, z)$ is constant. To see why this is so, let w be fixed and suppose z_1, z_2 are independent in $\mathcal{H}$. We have

$$\mu(w, z_1 + z_2)[U(z_1 + z_2) \otimes Ww] = \mu(w, z_1)[Uz_1 \otimes Ww] + \mu(w, z_2)[Uz_2 \otimes Ww].$$

From this, it follows that $\mu(w, z_1) = \mu(w, z_1 + z_2) = \mu(w, z_2)$, so that μ is independent of z. Similarly, it may be shown that μ is independent of w, from which we have that μ is constant. Since $\mu(e \otimes e) = 1$ we conclude that $\Phi(z \otimes w) = Uz \otimes Ww$. Using the adjoint of U we can write this last equation as

$$\Phi(z \otimes w) = W(z \otimes w)U^*. \tag{116}$$

If we suppose that (113) and (115) hold, then a completely analogous argument leads to

$$\Phi(z \otimes w) = V(z \otimes w)^* \tilde{Q}. \tag{117}$$

The proof of the lemma is completed by appealing to the density of the finite rank operators and these last two equations.

□

11.2.8. LEMMA. *If $0 < p < \infty$ and $p \neq 2$, then every surjective linear isometry Φ of $\mathcal{C}_p$ preserves rank.*

PROOF. It is sufficient to prove that $\Phi(\mathcal{R}) \subseteq \mathcal{R}$. We apply the conditions for equality in McCarthy's version of the Clarkson inequalities. We can state these as follows: for S and T in $\mathcal{C}_p$ with $0 < p < \infty$, $p \neq 2$,

$$\|S+T\|^p + \|S-T\|^p = 2(\|S\|^p + \|T\|^p) \tag{118}$$

if and only if $T^*S = TS^* = 0$. We observe that the condition is equivalent to $range(S) \perp range(T)$ and $range(S^*) \perp range(T^*)$. Thus equality cannot hold if $S+T$ has rank 1 unless $S = 0$ or $T = 0$.

To apply these conditions, suppose that $rank(A) = 1$ and $rank(\Phi(A)) > 1$. Write $\Phi(A) = \sum \mu_i(x_i \otimes y_i)$ where $\{x_i\}$ and $\{y_i\}$ are orthonormal sequences. Set $S = \mu_1(x_1 \otimes y_1)$ and $T = \sum_{i \geq 2} \mu_i(x_i \otimes y_i)$. For this choice of S and T it follows that

$$\|S \pm T\|^p = \sum \mu_i^p = \|S\|^p + \|T\|^p$$

and (118) holds. If we apply the inverse of Φ we get a similar decomposition of A and a contradiction. Therefore, if $rank(A) = 1$ it follows that $rank(\Phi(A)) = 1$ and this completes the proof of the lemma. □

The proof of Theorem 11.2.5 follows immediately from Lemmas 11.2.6, 11.2.7, and 11.2.8.

11.3. Isometries of Symmetric Norm Ideals: Sourour's Theorem

The primary focus of this section is on the results of Sourour concerning isometries of symmetric norm ideals of operators on $\mathcal{H}$. These results are very general and include the work of Arazy. The method of proof relies on the use of Hermitian operators on Banach spaces. We have illustrated these techniques in earlier chapters, but the results for function spaces do not apply in this situation and we need several lemmas.

If $A \in \mathcal{L}(\mathcal{H})$ and $(\mathcal{J}, \nu)$ is an ideal in $\mathcal{L}(\mathcal{H})$, then for every $T \in \mathcal{J}$, $TA \in \mathcal{J}$ and $AT \in \mathcal{J}$. These products clearly define linear maps on $\mathcal{J}$, and we see as a consequence of the following lemma that these maps are bounded.

11.3.1. LEMMA. *(Schatten) If $(\mathcal{J}, \nu)$ is a minimal norm ideal in $\mathcal{L}(\mathcal{H})$, then for any operator $A \in \mathcal{L}(\mathcal{H})$ and $T \in \mathcal{J}$ we have*

$$\nu(AT) \leq \|A\|\nu(T) \text{ and } \nu(TA) \leq \|A\|\nu(T).$$

PROOF. Let ν_0 denote the symmetric gauge function associated with ν and let $\lambda_1 \geq \lambda_2 \geq \cdots$ and $\mu_1 \geq \mu_2 \geq \cdots$ denote the eigenvalues of $|T|$ and

$|AT|$, respectively. Now $\{\lambda_n^2\}$ is the sequence of nonzero eigenvalues of T^*T and by a basic result of Fischer-Courant,

$$\lambda_n^2 = min\{max\langle T^*Tf, f\rangle\}$$

so that

$$\lambda_n = min\{max\|Tf\|\}.$$

Here the first max is taken over all f of norm 1 that are orthogonal to an $n-1$ dimensional subspace, and the minimum is over all such subspaces.

Similarly,

$$\mu_n = min\{max\|ATf\|\},$$

and we conclude that $\mu_n \leq \|A\|\lambda_n$ for each n. Therefore,

$$\begin{aligned}\nu(AT) &= \nu_0(AT) = \nu_0(\mu_1, \mu_2, \dots) \\ &\leq \nu_0(\|A\|\lambda_1, \|A\|\lambda_2, \dots) = \|A\|\nu_0(\lambda_1, \lambda_2, \dots) \\ &= \|A\|\nu_0(T) = \|A\|\nu(T).\end{aligned}$$

The other inequality follows by taking adjoints. □

If A and B are self-adjoint operators in $\mathcal{L}(\mathcal{H})$, and we define $\Phi : \mathcal{J} \to \mathcal{J}$ by $\Phi(T) = AT + TB$, then $e^{it\Phi}(T) = e^{itA}Te^{itB}$. It follows that e^{itA} and e^{itB} are unitaries on $\mathcal{H}$ so that $e^{it\Phi}$ is an isometry on $\mathcal{J}$ by property (ii) of Definition 11.1.1, and hence Φ is Hermitian on $(\mathcal{J}, \nu)$. The question arises as to whether every Hermitian operator Φ is of the form $\Phi(T) = AT + TB$ for some choice of self-adjoint operators A, B. This is in fact true and it is our purpose in this section to give an exposition of this result. We will assume throughout this section that $(\mathcal{J}, \nu)$ is a minimal norm ideal as given by Definition 11.1.1.

Our next lemma shows that Hermitian operators on $(\mathcal{J}, \nu)$ preserve a certain kind of orthogonality. We note that if, as usual, tr denotes the trace on the space $\mathcal{F}$, then the spaces $\mathcal{J}$ and $\mathcal{F}$ can be put in duality by letting $\langle\langle T, B\rangle\rangle = tr(B^*T)$, where $B \in \mathcal{F}$. (Recall that $\mathcal{F}$ denotes the space of finite rank operators.)

11.3.2. Lemma. *Suppose Φ is a Hermitian operator on $\mathcal{J}$. Let e_j and f_j be vectors in $\mathcal{H}$ for $j = 1, 2$ such that e_1 is orthogonal to e_2 and f_1 is orthogonal to f_2. If $T_j = f_j \otimes e_j$, then $\langle\langle \Phi(T_1), T_2\rangle\rangle = 0$.*

Proof. Let e_1, e_2 and f_1, f_2 be norm 1 vectors in $\mathcal{H}$ such that $e_1 \perp e_2$ and $f_1 \perp f_2$. Extend the sets $\{e_1, e_2\}$ and $\{f_1, f_2\}$ to orthonormal bases $\{e_j\}$ and $\{f_j\}$ for $\mathcal{H}$. Define the rank 1 operators $T_i = f_i \otimes e_i$. We introduce some new unitaries, U_j defined by $U_j e_j = -e_j$ while $U_j e_i = e_i$, for $i \neq j$. With these we define operators on $\mathcal{J}$ by $\Psi_j(T) = U_j T$ for $T \in \mathcal{J}$. It follows that $\Psi_j(T_i) = -T_j$ if $i = j$ while $\Psi_j(T_i) = T_i$ for $i \neq j$. Finally we introduce an operator Φ_0 on $\mathcal{J}$ by

$$\Phi_0 = \frac{1}{4}\{\Phi - \Psi_1^{-1}\Phi\Psi_1 - \Psi_2^{-1}\Phi\Psi_2 + \Psi_1^{-1}\Psi_2^{-1}\Phi\Psi_2\Psi_1\}.$$

Each of the Ψ_j is an invertible isometry on $\mathcal{J}$ by Definition 11.1.1 (ii), and so each term of the sum on the right above is Hermitian. Here, we have used the fact that conjugation of a Hermitian operator by an isometry and its inverse is still Hermitian. The operator Φ_0 is therefore a sum of Hermitian operators and is also Hermitian.

Let $a_{jk} = \langle\langle \Phi_0(T_j), T_k \rangle\rangle$. Using the fact that $\Psi_j^{-1} = \Psi_j$ and a simple property of the trace, a little computation shows that

$$a_{12} = \langle\langle \Phi(T_1), T_2 \rangle\rangle \text{ and } a_{21} = \langle\langle \Phi(T_2), T_1 \rangle\rangle$$

while $a_{jk} = 0$ for all other pairs j, k.

The norm ν when restricted to $\mathbb{R}^2$ is a standardized absolute norm in the sense of Schneider and Turner as discussed in Chapter 9. From Lemma 9.2.7, there exists a unit vector $\xi = (\xi_1, \xi_2)$ in $\mathbb{R}^2$ with *positive entries* such that, when viewed as a linear functional on $(\mathbb{R}^2, \nu)$, it is a support functional of the unit ball at ξ.

For $\lambda \in \mathbb{C}$, let $T = \xi_1 T_1 + \lambda \xi_2 T_2$. Then, the mapping $S \to \langle\langle S, T \rangle\rangle$ is a support functional of the unit ball in $\mathcal{J}$ at T. Since Φ_0 is Hermitian, $\langle\langle \Phi_0(T), T \rangle\rangle \in \mathbb{R}$. This requires that $\lambda a_{21} + \overline{\lambda} a_{12} \in \mathbb{R}$ for every choice of $\lambda \in \mathbb{C}$. Therefore $a_{12} = \overline{a_{21}}$.

Let n be the smallest positive integer for which ν is not the l^2 norm on $\mathbb{R}^n$. Given this fact, there must be a vector $\lambda = (\lambda_1, ..., \lambda_n)$ in $\mathbb{R}^n$ with $\nu(\lambda) = 1$ and a support functional $\mu = (\mu_1, ..., \mu_n)$ such that μ is not a scalar multiple of λ. We may assume without loss of generality that $\lambda_1 \mu_2 \neq \lambda_2 \mu_1$. Let $T = \sum_{j=1}^{n} \lambda_j \theta_j T_j$ and $S = \sum_{j=1}^{n} \mu_j \theta_j T_j$, where $|\theta_1| = 1$ and $\theta_j = 1$ for $j = 2, ..., n$. It is straightforward to show that S is a support functional at T. Since Φ_0 is Hermitian, $\langle\langle \Phi_0(T), S \rangle\rangle = \lambda_1 \mu_2 \theta_1 a_{12} + \lambda_2 \mu_1 \overline{\theta_1} \overline{a_{12}} \in \mathbb{R}$. Since this is true for all $\theta_1 \in \mathbb{C}$ with $|\theta_1| = 1$ and since $\lambda_1 \mu_2 \neq \lambda_2 \mu_1$, we conclude that $a_{12} = 0$. $\square$

11.3.3. Corollary. *If Φ is a Hermitian operator on $\mathcal{J}$, then for every pair of vectors e and f, there are vectors x and y such that*

$$\Phi(f \otimes e) = f \otimes x + y \otimes e.$$

Proof. Given vectors $e, f \in \mathcal{H}$, $\Phi(f \otimes e)$ is rank 1 so there are vectors $u, v \in \mathcal{H}$ such that $\Phi(f \otimes e) = v \otimes u$. If either of the pairs u, e or v, f is linearly dependent, it is easy to find x and y as desired. Hence let us suppose that both pairs are independent. Clearly, we may assume that both e and f have norm 1. Now we may take $e_1 = e, e_2 = u - \langle u, e \rangle e$ and $f_1 = f, f_2 = v - \langle v, f \rangle f$. If we apply Lemma 11.3.2 and use properties of the trace we conclude that either $\langle u, u \rangle = |\langle u, e \rangle|^2$ or $\langle v, v \rangle = |\langle v, f \rangle|^2$, which contradicts the independence. $\square$

The main theorem for Hermitian operators on $\mathcal{J}$ is the following.

11.3.4. Theorem. *(Sourour) Let $(\mathcal{J}, \nu)$ be a minimal norm ideal other than $\mathcal{C}_2$, and let Φ be a linear transformation on $\mathcal{J}$. Then Φ is a bounded*

Hermitian operator if and only if there are bounded self-adjoint operators A *and* B *on* $\mathcal{H}$ *such that* $\Phi(T) = AT + TB$ *for all* $T \in \mathcal{J}$.

PROOF. Let $\{e_j\}$ be an orthonormal basis for $\mathcal{H}$. By (11.3.3) there exist vectors f_j and g_j such that

$$\Phi(e_1 \otimes e_j) = g_j \otimes e_j + e_1 \otimes f_j.$$

Let $T_1 = e_1 \otimes (e_j + e_1)$ and $T_2 = e_k \otimes (e_j + e_1)$ for $k \neq 1$. Lemma 11.2.7 implies that $0 = \langle\langle \Phi(T_1), T_2 \rangle\rangle$, and some straightforward computation with the trace shows that $\langle\langle \Phi(T_1), T_2 \rangle\rangle = \langle e_k, g_1 - g_j \rangle$. Hence $(g_1 - g_j) \perp e_k$ for $k \neq 1$. Since the vectors $\{e_j\}$ form an orthonormal basis for $\mathcal{H}$, it follows that there exists a scalar λ_j such that $g_1 - g_j = \lambda_j e_1$. As a consequence we can write

$$\Phi(e_1 \otimes e_j) = e_1 \otimes f_j + g_j \otimes e_j = e_1 \otimes f_j' + g_1 \otimes e_j,$$

where $f_j' = f_j + \overline{\lambda_j} e_j$. We now define a linear transformation A_1 on $\mathcal{M}$, the linear span of the $\{e_k\}$, by $A_1 e_j = f_j'$. It follows immediately that for every $x \in \mathcal{M}$, $\Phi(e_1 \otimes x) = e_1 \otimes A_1 x + g_1 \otimes x$. The fact that Φ is bounded on $\mathcal{H}$ implies that A_1 is bounded on $\mathcal{M}$ and therefore extends to a bounded linear operator on $\mathcal{H}$.

If we repeat the same argument with e_k in place of e_1, we obtain the existence of a bounded operator A_k on $\mathcal{H}$ and a vector g_k' such that

$$\Phi(e_k \otimes x) = e_k \otimes A_k x + g_k' \otimes x.$$

We now show that $A_1 - A_j$ is in fact a scalar. Let $x, y \in \mathcal{H}$ with $x \perp y$. Set $T_1 = x \otimes (e_1 + e_j) \otimes x$ and $T_2 = y \otimes (e_1 - e_j) \otimes y$. Lemma 11.2.7 applies and we conclude that

$$\langle A_1 x - A_j x, y \rangle = \langle\langle \Phi(T_1), T_2 \rangle\rangle = 0.$$

Thus every vector is an eigenvector of $A - A_j$ and so $A - A_j$ must be a scalar multiple of the identity. Hence, there exist vectors h_j in $\mathcal{H}$ such that

$$\Phi(e_j \otimes x) = e_j \otimes Ax + h_j \otimes x,$$

where $A = A_1$. Just as in the preceding case we define an operator B^* on the span $\mathcal{M}$ by $B^* e_j = h_j$. This operator extends to a bounded operator on all of $\mathcal{H}$ and

$$\Phi(y \otimes x) = y \otimes Ax + B^* y \otimes x = A(y \otimes x) + (y \otimes x)B$$

for every $x, y \in \mathcal{H}$. The same formula holds for every finite rank operator on $\mathcal{H}$, and by density of the finite rank operators $\mathcal{F}$ it follows that

$$\Phi(T) = AT + TB$$

for every $T \in \mathcal{H}$. To complete the proof we must show that the operators A and B are self-adjoint. To that end we suppose that $A = A_1 + iA_2$ and $B = B_1 + iB_2$ are the cartesian decompositions of A and B. The Hermitian operator Φ can now be written as $\Phi(T) = \Phi_1(T) + i\Phi_2(T)$ where $\Phi_j(T) = A_j T + iTB_j$. Both Φ_1 and Φ_2 are Hermitian and as a consequence $i\Phi_2 = \Phi - \Phi_1$ must also be Hermitian. But this now implies that Φ_2 and $i\Phi_2$ are both Hermitian. An application of Liouville's theorem to the vector-valued function $f(z) = e^{z\Phi_2}$ implies that $\Phi_2 = 0$. Therefore

$$\Phi(T) = \Phi_1(T) = A_1 T + T B_1,$$

where A_1 and B_1 are self-adjoint members of $\mathcal{L}(\mathcal{H})$. This completes the proof. □

11.3.5. LEMMA. *Let A, B, C, D be bounded operators on H and assume that*

$$ATB = CT + TD$$

for every finite rank operator T. Then either A and C are scalars or else B and D are scalars.

PROOF. We claim that either C or D is a scalar multiple of the identity. If this were not true we could find vectors f and g in $\mathcal{H}$ such that the sets $\{f, Cf\}$, $\{g, D^*g\}$ are linearly independent. Set $T = g \otimes f$. Then $CT + TD = g \otimes Cf + D^*g \otimes f$ is an operator of rank 2. But the operator ATB can have rank at most 1. This contradicts the statement $AXB = CX + XB$. Therefore either C or D is a scalar.

If we suppose that D is scalar, then without loss of generality we may assume that $D = 0$. Thus we assume that $ATB = CT$. If we set $T = f \otimes e$, then it follows that $B^*f \otimes Ae = f \otimes Ce$, and hence for every vector $u \in \mathcal{H}$,

$$\langle u, B^*f\rangle\langle Ae, e\rangle = \langle u, f\rangle\langle Ce, e\rangle.$$

Clearly then $A = C = 0$ or there exists a scalar λ such that $Bf = \lambda f$ for every $f \in \mathcal{H}$. The case that C is a scalar is treated in a similar manner. □

11.3.6. COROLLARY. *Let Φ be a bounded operator on $\mathcal{J}$. Then Φ and Φ^2 are both Hermitian operators if and only if Φ is either left multiplication or right multiplication by a self-adjoint operator on $\mathcal{H}$.*

PROOF. If there exists a self-adjoint element A of $\mathcal{L}(\mathcal{H})$ such that $\Phi(T) = AT$, then it is clear that Φ and Φ^2 are both Hermitian. On the other hand, if both Φ and Φ^2 are Hermitian, by Theorem 11.3.4 there exist self-adjoint elements A and B of $\mathcal{L}(\mathcal{H})$ such that $\Phi(T) = AT + TB$, and there exist self-adjoint elements C and D of $\mathcal{L}(\mathcal{H})$ such that

$$A^2T + 2ATB + TB^2 = CT + TD.$$

The preceding lemma implies that either A or B is scalar. □

We are now ready to consider isometries of $(\mathcal{J}, \nu)$.

11.3.7. THEOREM. *(Sourour) Let $(\mathcal{J}, \nu)$ be a minimal norm ideal other than $\mathcal{C}_2$, and let Φ be a linear transformation on $\mathcal{J}$. Then Φ is a surjective isometry of $\mathcal{J}$ if and only if there are unitary operators U and V on $\mathcal{H}$ such that*

$$\Phi(T) = UTV \quad \text{or} \quad \Phi(T) = UT^tV$$

for every $T \in \mathcal{J}$.

PROOF. Let Φ be a surjective isometry of $(\mathcal{J}, \nu)$. Given $A \in \mathcal{L}(\mathcal{H})$ define the left (respectively right) multiplication operators L_A and R_A on $\mathcal{J}$ by $L_A(T) = AT$ and $R_A(T) = TA$ for $T \in \mathcal{J}$. For every self-adjoint member A of $\mathcal{L}(\mathcal{H})$, L_A and L_A^2 are Hermitian operators on $\mathcal{J}$. Since Φ is a surjective isometry it follows that $\Phi L_A \Phi^{-1}$ and $\Phi L_A^2 \Phi^{-1}$ are also Hermitian. Since $\Phi L_A^2 \Phi^{-1} = (\Phi L_A \Phi^{-1})^2$ it follows from 11.3.6 that $\Phi L_A \Phi^{-1}$ is either a left multiplication or a right multiplication by some self-adjoint element of $\mathcal{L}(\mathcal{H})$. We claim that $\Phi L_A \Phi^{-1}$ is always either a left multiplication or a right multiplication independent of the choice of the operator A. To see this, suppose that $\Phi L_B \Phi^{-1} = L_S$ while $\Phi L_C \Phi^{-1} = R_T$. Then $\Phi L_{(B+C)} \Phi^{-1} = L_S + R_T$. For the operator $L_S + R_T$ to be either a left multiplication or a right multiplication it follows from the arguments in Lemma 11.3.5 that one of the operators S or T must be a multiple of the identity and so $\Phi L_A \Phi^{-1}$ is always a left multiplication or a right multiplication. Since every element of $\mathcal{L}(\mathcal{H})$ has a cartesian decomposition, it follows that $\Phi L_T \Phi^{-1}$ is also always a left or right multiplication.

We make further reduction. We claim that without loss of generality we may assume that $\Phi L_A \Phi^{-1}$ is always a left multiplication. For suppose that $\Phi L_A \Phi^{-1} = R_C(T)$. In this case, let $\xi(T) = T^t$ and $\Phi_1 = \xi\Phi$. Then Φ_1 is an isometry and $\Phi_1 L_A \Phi_1^{-1}(T) = L_{C^t}(T)$. Likewise, we may assume that $\Phi R_A \Phi^{-1}$ is always a right multiplication.

We can now proceed with the remainder of the proof. We define a map $\psi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ by $\Phi L_A \Phi^{-1} = L_{\psi(A)}$. The map ψ has the property that $(\psi(A))^* = \psi(A^*)$ and hence ψ is a *-automorphism of $\mathcal{L}(\mathcal{H})$. As we have seen before, there exists a unitary operator U such that $\psi(A) = UAU^*$ for every $A \in \mathcal{L}(\mathcal{H})$. Therefore $\Phi L_A \Phi^{-1} = L_{UAU^*}$ for every $A \in \mathcal{L}(\mathcal{H})$. A completely analogous argument gives rise to another unitary V such that $\Phi R_A \Phi^{-1} = R_{V^*AV}$.

We use these unitaries to define an isometry $\Phi_0 : \mathcal{J} \to \mathcal{J}$ by $\Phi_0(T) = U^*\Phi(T)V^*$. We claim that Φ_0 is a scalar multiple of the identity. It is easy to show that $\Phi_0(L_A R_B T) = L_A R_B \Phi_0(T)$ for every $T \in \mathcal{J}$ and $A,\ B \in \mathcal{L}(\mathcal{H})$. Hence Φ_0 commutes with left and right multiplications. In particular, if $e, f \in \mathcal{H}$ and E and F denote the projections onto the one-dimensional subspaces spanned by e and f, respectively, then Φ_0 commutes with the product $L_E R_F$. Since Φ_0 preserves rank 1 operators, it follows that there must exist a scalar λ such that $\Phi_0(f \otimes e) = f \otimes \lambda e$. We claim that this scalar does not depend on the particular choice of e and f. To see that, suppose that $\Phi_0(f_1 \otimes e_1) = f_1 \otimes \lambda_1 e_1$ and $\Phi_0(f_2 \otimes e_2) = f_2 \otimes \lambda_2 e_2$. Let A and B be members of $\mathcal{L}(\mathcal{H})$ such that $Ae_1 = e_2$ and $Be_2 = e_1$. Since Φ_0 commutes with $L_A R_B$, a simple computation shows that $\lambda_1 = \lambda_2$. We have shown that there exists λ such that $\Phi_0(T) = \lambda T$ for all $T \in \mathbb{R}$ and by density of the finite ranks this is true for all $T \in \mathcal{J}$. Therefore, $\Phi(T) = U(\lambda T)V$ for all $T \in \mathcal{J}$. Since $|\lambda| = 1$, it can absorbed into the unitary U and the proof is finished.

The proof in the other direction follows from the definition of a symmetric norm and the fact that for any compact operator T, the s-numbers of T^t are the same as those for T.

□

11.4. Noncommutative L^p Spaces

The Schatten classes considered in Section 2 could be thought of as noncommutative L^p spaces (as well as ideals) and in this section we want to examine a more general notion of L^p spaces. If $\mathcal{M}$ is a von Neumann algebra, (that is, a C^*-algebra of operators on a Hilbert space $\mathcal{H}$ that is closed in the weak operator topology), a *trace* on $\mathcal{M}$ is a nonnegative extended real-valued map τ on the positive part $\mathcal{M}_+$ of $\mathcal{M}$ which satisfies

(i) $\tau(S+T) = \tau(S) + \tau(T)$, for all $T, S \in \mathcal{M}_+$;
(ii) $\tau(rS) = r\tau(S)$, for all $r \geq 0$ and $S \in \mathcal{M}_+$;
(iii) $\tau(U^*U) = \tau(UU^*)$, for all $U \in \mathcal{M}$.

We say that τ is *normal* if $\sup \tau(S_\alpha) = \tau(\sup S_\alpha)$ for any bounded increasing net $\{S_\alpha\}$ in $\mathcal{M}_+$, *semifinite* if for any nonzero $S \in \mathcal{M}_+$ there is a nonzero $T \in \mathcal{M}_+$ such that $T \leq S$ and $\tau(T) < \infty$, and *faithful* if $\tau(S) = 0$ implies $S = 0$. If $\tau(I) < \infty$, we say that τ is *finite*. (Here, I denotes the identity of $\mathcal{M}$.) A von Neumann algebra $\mathcal{M}$ is said to be *semifinite* if it admits a normal semifinite faithful trace.

11.4.1. DEFINITION. *Let $\mathcal{M}$ denote a semifinite von Neumann algebra with a given normal semifinite faithful trace τ. For $T \in \mathcal{M}$ and $1 \leq p < \infty$, let*

$$\|T\|_p = \tau(|T|^p)^{1/p},$$

where, as usual, $|T|$ denotes the operator $(T^*T)^{1/2}$. Then $\|\cdot\|$ defines a norm on the set of T for which it is finite, and we let $L^p(\mathcal{M}, \tau)$ denote the norm completion of that set. The space $L^p(\mathcal{M}, \tau)$ is called the noncommutative L^p space associated with (or obtained from) $(\mathcal{M}, \tau)$. For $p = \infty$, we identify $L^\infty(\mathcal{M}, \tau)$ with $\mathcal{M}$ given the operator norm.

It is possible to identify elements of $L^p(\mathcal{M}, \tau)$ with certain τ-measurable (but not necessarily bounded) operators affiliated with $\mathcal{M}$. An operator S defined on the Hilbert space $\mathcal{H}$ is said to be *affiliated* with $\mathcal{M}$ if $SU = US$ for any unitary U in the commutant $\mathcal{M}'$ of $\mathcal{M}$.

A considerable amount of literature exists in which properties of the noncommutative L^p spaces are developed, but we intend to discuss as little of that as possible here. We will give some references in the notes at the end of the chapter. We do wish to point out that the classical L^p spaces can be also regarded as associated with (commutative) von Neumann algebras. Suppose we consider the usual Lebesgue measure space $([0,1], \Sigma, \lambda)$ and let $\mathcal{H} = L^2([0,1], \Sigma, \lambda)$. Let $\mathcal{M}$ denote the multiplication algebra of $([0,1], \Sigma, \lambda)$, that is the algebra of all operators T of the form $Tf(t) = k(t)f(t)$ for $f \in \mathcal{H}$, and where k is a bounded measurable function on $([0,1],\Sigma,\lambda)$. For every projection P in $\mathcal{M}$ there is a measurable set E such that P is the operation of multiplication by the characteristic function of E. If we let $\tau(P)=\lambda(E)$, τ can be extended to all of $\mathcal{M}_+$ so that it has the desired properties of a trace. Thus $L^p(\mathcal{M},\tau)$ is really just $L^p([0,1],\Sigma,\lambda)$, but $\mathcal{M}$ is commutative. Indeed, one could just define the trace directly by letting it simply send the operator of multiplication by f to the integral $\int f d\lambda$ and avoid the mention of projections.

Just as Kadison's theorem on isometries of C^*-algebras can be thought of as a noncommutative version of the Banach-Stone theorem, we might wonder if there is a noncommutative version of Lamperti's theorem characterizing the isometries on the classical L^p spaces. We recall (Chapter 3, Theorem 3.2.5) that a linear isometry U from $L^p(\Omega_1,\Sigma_1,\mu_1)$ into $L^p(\Omega_2,\Sigma_2,\mu_2)$ is of the form

$$Uf(t)=h(t)T_1f(t)$$

where T_1 is a transformation induced by a regular set isomorphism T and h is a function whose pth power is related to the Radon-Nikodym derivative of $(\mu_1\circ T^{-1})$ with respect to μ_2. (See also Chapter 8.) Note that a regular set isomorphism can be thought of as a map on the projections in the associated L^∞ algebra, and so may suggest something for the noncommutative case.

These ideas go back a long way and one of the earlier published results is due to Russo and deals with isometries on a noncommutative L^1 space for finite traces. We will give that theorem first, and then we will talk about Yeadon's results for the semifinite case of noncommutative L^p spaces. Much more has been done, but we will restrict ourselves to giving a brief discussion of that in the notes. Observe that the isometries on $L^\infty(\mathcal{M},\tau)$ have already been described by Kadison. We also note that $L^1(\mathcal{M},\tau)$ is the predual of $\mathcal{M}=L^\infty(\mathcal{M},\tau)$. In fact, it is known that the usual L^p, L^q dualities occur where $1/p+1/q=1$, and are given by $\langle\langle T,S\rangle\rangle=\tau(TS)$, and in the special case of $p=2$, the inner product for $L^2(\mathcal{M},\tau)$ is given by $\langle T,S\rangle=\tau(S^*T)$.

11.4.2. Theorem. *(Russo) Let $\mathcal{M}$ be a von Neumann algebra with a faithful finite normal trace τ and let Φ be a linear isometry of $L^1(\mathcal{M},\tau)$ onto itself. Then there is a Jordan*-isomorphism α of $\mathcal{M}$, a positive operator $V\in L^2(\mathcal{M},\tau)$ affiliated with the center $\mathcal{Z}$ of $\mathcal{M}$, and a unitary operator U in $\mathcal{M}$ such that*

$$\Phi(T)=\alpha(T)V^2U, \text{ for all } T\in\mathcal{M}.$$

Proof. The adjoint $(\Phi^{-1})^*$ of Φ^{-1} (a Banach space adjoint) is an isometry of $L^\infty(\mathcal{M},\tau)$, and so by Kadison's theorem, there is a Jordan*-isomorphism α and a unitary operator $W\in\mathcal{M}$ such that

$$(\Phi^{-1})^*(T)=W\alpha(T) \text{ for all } T\in\mathcal{M}.$$

There is an isometry Ψ of $L^1(\mathcal{M}, \tau)$ such that $\Psi^* = \alpha$. Hence, if $T, S \in \mathcal{M}$, we have

$$\begin{aligned}\langle\langle \Phi^{-1}(T), S\rangle\rangle &= \langle\langle T, (\Phi^{-1})^*(S)\rangle\rangle \\ &= \langle\langle T, W\alpha(S)\rangle\rangle = \langle\langle TW, \alpha(S)\rangle\rangle = \langle\langle \Psi(TW), S\rangle\rangle.\end{aligned}$$

Then it follows that

$$\Phi^{-1}(T) = \Psi(TW). \tag{119}$$

By a theorem of Broise [**55**] there exists a positive operator V affiliated with the center $\mathcal{Z}$ of $\mathcal{M}$ such that the map $T \to V\alpha(T)$ is an L^2 isometry on $\mathcal{M}$. Hence for $T \in \mathcal{M}$, both V and $\alpha(T)V$ are in $L^2(\mathcal{M}, \tau)$ so by Holder's inequality (which is known to hold for these spaces) it must be the case that $\alpha(T)V^2$ belongs to $L^1(\mathcal{M}, \tau)$. If $V \in \mathcal{M}$ it is straightforward to show that

$$\langle\langle \alpha(T)V^2, \alpha(S)\rangle\rangle = \langle V\alpha(S), V\alpha(T^*)\rangle. \tag{120}$$

If V is not in $\mathcal{M}$, write $V = \int_0^\infty \lambda de_\lambda$ where $e_\lambda \in \mathcal{Z}$ so that for each positive integer n, $V_n = \int_0^n \lambda de_\lambda \in \mathcal{Z}$. It can be shown that $V_nV = VV_n$ and $\|V_n - V\|_2 \to 0$. Therefore,

$$\|V^2 - V_m^2\|_1 = \|(V + V_m)(V - V_m)\|_1 \leq \|V + V_m\|_2\|V - V_m\|_2 \to 0.$$

Hence

$$\begin{aligned}\langle\langle \alpha(T)V^2, \alpha(S)\rangle\rangle &= \lim_n \langle\langle \alpha(T)V_n^2, \alpha(S)\rangle\rangle \\ &= \lim_n \langle V_n\alpha(S), V_n\alpha(T^*)\rangle \\ &= \langle V\alpha(S), V\alpha(T^*)\rangle,\end{aligned}$$

which establishes (120) for all V. Recalling that $\Psi^* = \alpha$ and that $V\alpha$ is an L^2 isometry, we obtain, for $T, S \in \mathcal{M}$,

$$\begin{aligned}\langle\langle \Psi(\alpha(T)V^2, S\rangle\rangle &= \langle\langle \alpha(T)V^2, \alpha(S)\rangle\rangle \\ &= \langle V\alpha(S), V\alpha(T^*)\rangle \\ &= \langle S, T^*\rangle = \langle\langle T, S\rangle\rangle,\end{aligned}$$

so that $\Psi(\alpha(T)V^2) = T$. Now putting this together with (119), we have

$$T = \Psi(\alpha(T)V^2W^{-1}W) = \Phi^{-1}(\alpha(T)V^2W^{-1}).$$

Thus $\Phi(T) = \alpha(T)V^2W^{-1}$ and the proof is complete. □

To characterize the isometries for the general case of $L^p(\mathcal{M}, \tau)$, as was also the case for the classical L^p spaces and for the $\mathcal{C}_p$ spaces, we will need a version of Clarkson's inequalities and, in particular, conditions under which equality holds. We state, without proof, a version as given by Yeadon for the case of a semifinite trace.

11.4.3. THEOREM. *Let τ be a faithful semifinite normal trace on a von Neumann algebra $\mathcal{M}$, and let $S, T \in L^p(\mathcal{M}, \tau)$ where $1 \leq p < \infty, p \neq 2$. Then equality*

$$\|S+T\|_p^p + \|S-T\|_p^p = 2(\|S\|_p^p + \|T\|_p^p)$$

holds in Clarkson's inequality

$$\|S+T\|_p^p + \|S-T\|_p^p \leq 2(\|S\|_p^p + \|T\|_p^p) \ \ (1 \leq p < 2),$$

$$\|S+T\|_p^p + \|S-T\|_p^p \geq 2(\|S\|_p^p + \|T\|_p^p) \ \ (2 < p < \infty),$$

if and only if $ST^* = S^*T = 0$.

We will now state and give a sketch of the proof of Yeadon's theorem concerning isometries on $L^p(\mathcal{M}, \tau)$ spaces with semifinite trace. In this proof, the notation $s(T)$ will denote the projection (called the *support projection* of T) W^*W, where W is the partial isometry in the polar decomposition $T = W|T|$ of T. An operator $T \in \mathcal{M}$ will be called an *elementary operator* if $\tau(s(T)) < \infty$, and by $\mathcal{E}$ we will mean the set of elementary operators.

11.4.4. THEOREM. *(Yeadon) For $i = 1, 2$, let τ_i be a faithful semifinite normal trace on a von Neumann algebra $\mathcal{M}_i$. A continuous linear operator $\Phi : L^p(\mathcal{M}_1, \tau_1) \to L^p(\mathcal{M}_2, \tau_2)$ (for $1 \leq p < \infty, p \neq 2$) is an isometry if and only if there exist, uniquely, a partial isometry $W \in \mathcal{M}_2$, a positive self-adjoint operator B affiliated with $\mathcal{M}_2$, and a Jordan*-isomorphism α of $\mathcal{M}_1$ onto a weakly closed $*$-subalgebra of $\mathcal{M}_2$ such that*

(i) *$W^*W = \alpha(I) = s(B)$;*
(ii) *every spectral projection of B commutes with $\alpha(T)$ for all $T \in \mathcal{M}_1$;*
(iii) *$\tau_1(T) = \tau_2(B^p\alpha(T))$ for all $T \in \mathcal{M}_1, T \geq 0$; and*
(iv) *$\Phi(T) = WB\alpha(T)$ for all $T \in L^p(\mathcal{M}_1, \tau_1) \cap \mathcal{M}_1$.*

We note that the form in this theorem is in reverse order to the form in Russo's theorem, but one can be transformed to the other by means of the $*$-operation.

PROOF. Suppose that α, B, and W are as given in the statement of the theorem above, and that Φ is given by $\Phi(T) = WB\alpha(T)$ for $T \in L^p(\mathcal{M}_1, \tau_1) \cap \mathcal{M}_1$. Using Kadison's result that a Jordan*-isomorphism can be written as a direct sum of a *-isomorphism and a *-anti-isomorphism, let P be the central projection in $\mathcal{M}_1$ such that α restricted to $\mathcal{M}_1 P$ is a *-isomorphism and α restricted to $\mathcal{M}_1(I-P)$ is a *-anti-isomorphism. Hence

$$|\Phi(T)| = B|\alpha(T)| = B\alpha(P|T| + (I-P)|T^*|),$$

$$\begin{aligned}|\Phi(T)|^p &= B^p(\alpha(P|T| + (I-P)|T^*|)^p = B^p\alpha((P|T| + (I-P)|T^*|)^p) \\ &= B^p\alpha(P|T| + (I-P)|T^*|^p),\end{aligned}$$

and

$$\begin{aligned}\tau_2(|\Phi(T)|^p) &= \tau_2(B^p\alpha(P|T|^p + (I-P)|T^*|^p)) \\ &= \tau_1(P|T|^p + (I-P)|T^*|^p) \\ &= \tau_1(|T|^p)\end{aligned}$$

so that Φ is an isometry.

For the converse, suppose that Φ is an isometry, and if E is a projection in $\mathcal{E}_1$, the elementary operators in $\mathcal{M}_1$, let $\Phi(E) = W_E B_E$ be the polar decomposition of $\Phi(E)$ in $\mathcal{M}_2$. If E, F are two such projections with $EF = 0$, then

$$\|E \pm F\|_p^p = \|E\|_p^p + \|F\|_p^p, \quad \|\Phi(E) \pm \Phi(F)\|_p^p = \|\Phi(E)\|_p^p + \|\Phi(F)\|_p^p,$$

and by Theorem 11.4.3 we have $\Phi(E)\Phi(F)^* = \Phi(E)^*\Phi(F) = 0$, or equivalently

$$W_E W_F^* = W_E^* W_F = 0.$$

From this, it follows that $W_{E+F} = W_E + W_F$ and $B_{E+F} = B_E + B_F$. Thus, if we define α on the projections by $\alpha(E) = W_E^* W_E$, it is clear that α is additive. Now for a self-adjoint simple operator $T = \sum_{j=1}^n \lambda_j E_j \in \mathcal{M}_1$, where the λ_j are real and the projections $E_j \in \mathcal{E}_1$ are pairwise orthogonal, we extend the definition of α by $\alpha(T) = \sum_{j=1}^n \lambda_j \alpha(E_j)$. Then α can be shown to be square preserving, isometric in the $\|\cdot\|_\infty$ norm, and real linear on the commuting self-adjoint simple operators in $\mathcal{M}_1$. If $T = T^* \in \mathcal{E}_1$, and $\{f_n(\lambda)\}$ is a sequence of step functions which are zero at zero and converge uniformly to λ on the spectrum of T, we define $\alpha(T)$ to be the $\|\cdot\|_\infty$ limit of the simple operators $\alpha(f_n(T))$ in $\mathcal{M}_2$.

If E, F are projections in $\mathcal{E}_1$ with $F \leq E$, then because $\Phi(F)\alpha(F) = \Phi(F)$ and $\Phi(E-F)\alpha(F) = 0$, we have $\Phi(F) = \Phi(E)\alpha(F)$, so that $\Phi(T) = \Phi(E)\alpha(T)$ when T is a self-adjoint simple operator with $s(T) \leq E$. For such T, E, and f_n the following holds:

$$\|\Phi(E)(\alpha(T) - \alpha(f_n(T)))\|_p \leq \|\Phi(E)\|_p \|\alpha(X) - \alpha(f_n(T))\|_\infty.$$

Thus, we have

$$\Phi(E)\alpha(T) = \lim_{p,n} \Phi(E)\alpha(f_n(T)) = \lim_{p,n} \Phi(f_n(T)).$$

(By the symbol $\lim_{p,n}$ we mean the limit in the $\|\cdot\|_p$-norm as $n \to \infty$.) Since $T = \lim_{p,n} f_n(T)$, it follows that

$$\Phi(T) = \lim_{p,n} \Phi(f_n(T)) = \Phi(E)\alpha(T).$$

Now if $T, S \in \mathcal{E}_1$ are self-adjoint, $E = s(T) \vee s(S) \in \mathcal{E}_1$ and

$$\Phi(E)(\alpha(T+S) - \alpha(T) - \alpha(S)) = \Phi(T+S) - \Phi(T) - \Phi(S) = 0.$$

Since $\alpha(T+S) - \alpha(T) - \alpha(S)$ has range projection contained in the support projection $\alpha(E)$ of $\Phi(T)$, it follows that $\alpha(T+S) - \alpha(T) - \alpha(S) = 0$. Therefore, α as defined so far is real linear and the properties $\alpha(T^2) = \alpha(T)^2$ and

$\|\alpha(T)\|_\infty = \|T\|_\infty$ continue to hold by $\|\cdot\|_\infty$ continuity. We can extend α to a $\|\cdot\|_\infty$-continuous complex-linear map by defining

$$\alpha(T + iS) = \alpha(T) + i\alpha(S)$$

for S, T self-adjoint, and we have $\alpha(T^*) = \alpha(T)^*$ and $\alpha(T^2) = \alpha(T)^2$ for non-self-adjoint $T \in \mathcal{E}_1$. Furthermore, given projections $E, F \in \mathcal{E}_1$ with $F \leq E$ we have $B_{E-F}\alpha(F) = 0$ and $B_F\alpha(F) = B_F$ from which we conclude that $B_E\alpha(F) = B_F = \alpha(F)B_E$. Thus, B_E commutes with $\alpha(T)$ for each self-adjoint simple operator T with support contained in E. Therefore, if a self-adjoint operator $T \in \mathcal{E}_1$ with support contained in E is the $\|\cdot\|_\infty$ limit of simple operators T_n we have

$$B_E\alpha(T) = \lim_{p,n} B_E\alpha(T_n) = \lim_{p,n} \alpha(T_n)B_E = \alpha(T)B_E.$$

Moreover, if $E \geq F$ as above, then

$$\tau_1(F) = \|F\|_p^p = \|\Phi(F)\|_p^p = \tau_2(B_E^p\alpha(F)),$$

and so by linearity and continuity,

$$\tau_1(T) = \tau_2(B_E^p\alpha(T))$$

for $T \in E\mathcal{M}_1E$. The normality of τ_1 and τ_2 implies that the restriction of α to $E\mathcal{M}_1E$ is normal, and this completes the proof when $\tau_1(I) < \infty$, taking $W = W_I$ and $B = B_I$.

If $\tau_1(I) = \infty$, we extend α to all of $\mathcal{M}_1$ by defining $\alpha(T)$ as the limit of the $\|\cdot\|$-bounded net $\alpha(ETE)$ indexed by the upwards directed set of projections $E \in \mathcal{E}_1$. For each $T \in \mathcal{M}_1$, the net $\alpha(ETE)$ converges in the strong operator topology since

$$\alpha(FTF) = \alpha(F)\alpha(ETE)\alpha(F)$$

when $F \leq E \in \mathcal{E}_1$, and the union of the ranges of $\alpha(F)$ for $F \in \mathcal{E}_1$ is dense in the range of $\alpha(I) = \sup\{\alpha(F) : F \in \mathcal{E}_1\}$. Hence, $\alpha(ETE) = \alpha(E)\alpha(T)\alpha(E)$ if $T \in \mathcal{M}_1$ and $E \in \mathcal{E}_1$. Similarly we can define W as the strong operator limit of W_E as $E \to I$, since $W_F = W_E\alpha(F)$ if $F \leq E \in \mathcal{E}_1$, and then $W_E = W\alpha(E)$ for $E \in \mathcal{E}_1$.

It can be seen that α is linear and also normal, since, if $\{T_\beta\}$ is a bounded monotone net of self-adjoint elements of $\mathcal{M}_1$, then $\{\alpha(T_\beta)\}$ is a similar net in $\mathcal{M}_2$, and if $T = \lim T_\beta$, then for each $E \in \mathcal{E}_1$,

$$\begin{aligned}\alpha(E)\alpha(T)\alpha(E) = \alpha(ETE) &= \lim \alpha(ET_\beta E) = \lim \alpha(E)\alpha(T_\beta)\alpha(E)\\ &= \alpha(E)\lim \alpha(T_\beta)\alpha(E),\end{aligned}$$

so that $\alpha(T) = \lim \alpha(T_\beta)$. Thus for self-adjoint $T \in \mathcal{M}_1$, we have

$$\begin{aligned}\alpha(T^2) &= \lim \alpha(TET) = \lim \alpha(E)\alpha(TET)\alpha(E) = \lim \alpha(ETETE)\\ &= \lim(\alpha(ETE))^2 = \alpha(T)^2.\end{aligned}$$

The operator B is most easily defined by its spectral projections. For each projection $E \in \mathcal{E}_1$ let

$$B_E = \int_0^\infty \lambda dP_E(\lambda)$$

be the spectral resolution, so that $\alpha(E) = I - P_E(0)$. Now, since $B_F = B\alpha(E)$ if $F \leq E$, we have, for $\lambda \geq 0$, $I - P_F(\lambda) = \alpha(F)(I - P(\lambda))$ and we can define $P(\lambda)$ to be the strong operator limit of $P_E(\lambda)$ as E increases to I, and then define

$$B = \int_0^\infty \lambda dP(\lambda).$$

Next, we see that if $E \in \mathcal{E}_1$, we have $I - P_E(\lambda) = \alpha(E)(I - P(\lambda))$ and $B_E = B\alpha(E)$. Hence, each $P(\lambda)$ commutes with $\alpha(E)$ if $E \in \mathcal{E}_1$ and therefore commutes with $\alpha(T)$ for each $T \in \mathcal{M}_1$.

Suppose $T \in \mathcal{M}_1, T \geq 0, \tau_1(T) < \infty$ and let $\{E_n\}$ be an increasing sequence of spectral projections of T, each with finite trace, and $\lim E_n = s(T)$. Then the $\alpha(E_n)$ are spectral projections of $\alpha(T)$, and

$$\begin{aligned}\tau_2(B^p\alpha(T)\alpha(E_n)) &= \tau_2(B^p\alpha(E_n)\alpha(T)\alpha(E_n)) = \tau_2(B^p_{E_n}\alpha(E_nTE_n)) \\ &= \tau_1(E_nTE_n),\end{aligned}$$

which goes to $\tau_1(T)$ as $n \to \infty$. Since B^p is positive self-adjoint and commutes with the bounded positive operator $\alpha(T)$, the operator $B^p\alpha(T)$ is positive self-adjoint and has a spectral resolution

$$B^p\alpha(T) = \int_0^\infty \lambda dQ(\lambda).$$

Since $B^p\alpha(T)$ commutes with $\alpha(E_n)$, we have

$$B^p\alpha(T)\alpha(E_n) = \int_0^\infty \lambda dQ_n(\lambda),$$

where $(I - Q_n(\lambda)) = (I - Q(\lambda))\alpha(E_n)$, which implies, for fixed $\lambda \geq 0$, that $I - Q_n(\lambda)$ increases to $I - Q(\lambda)$ as $n \to \infty$. Thus, since

$$\tau_2(B^p\alpha(T)\alpha(E_n) = \int_0^\infty \tau_2(I - Q_n(\lambda))d\lambda,$$

we have, by monotone convergence,

$$\tau_1(T) = \int_0^\infty \tau_2(I - Q(\lambda))d\lambda = \tau_2(B^p\alpha(T)).$$

If $T \in \mathcal{M}_1, T \geq 0$, and $\tau_1(T) = \infty$, we can find $E \in \mathcal{E}_1$ with $\tau_1(ETE)$ arbitrarily large. Then

$$\begin{aligned}\tau_2(\alpha(E)B^p\alpha(T)\alpha(E)) &= \tau_2(B^p\alpha(E)\alpha(T)\alpha(E)) \\ &= \tau_2(B^p_E\alpha(ETE)) = \tau_1(ETE),\end{aligned}$$

and so $\tau_2(B^p\alpha(T)) = \infty$.

To complete the proof, suppose that $T \in L^p(\mathcal{M}_1, \tau_1) \cap \mathcal{M}_1$. Then by the argument given at the beginning of the proof,

$$|B\alpha(T)| = B|\alpha(T)| \text{ and } \tau_2(|B\alpha(T)|^p) = \tau_2(B^p|\alpha(T)|^p) = \tau_1(|T|^p),$$

so that the map $T \to WB\alpha(T)$ is isometric from $L^p(\mathcal{M}_1, \tau_1) \cap \mathcal{M}_1$ into $L^p(\mathcal{M}_2, \tau_2)$. If $T \in \mathcal{E}_1$ and $E = s(T) \vee s(T^*)$, then

$$\begin{aligned}\Phi(T) &= W_E B_E \alpha(T) = W\alpha(E)B_E\alpha(T) = WB\alpha(E)\alpha(T) \\ &= WB\alpha(T),\end{aligned}$$

and so

$$\Phi(T) = WB\alpha(T)$$

holds for all $T \in L^p(\mathcal{M}_1, \tau_1) \cap \mathcal{M}_1$.

We omit the argument about uniqueness.

□

11.5. Notes and Remarks

The definition of a symmetric norm ideal was taken from the paper of Sourour [**355**], who notes that condition (iii) in Definition 11.1.1 is equivalent to the separability of $\mathcal{J}$. Good references for a general treatment of these ideas are [**146**] and [**335**]. According to Gohberg and Krein [**146**, p. 24], the s-numbers were first introduced by Schmidt in the study of integral equations with nonsymmetric kernels.

Isometries of $\mathbb{C}_p$. The study of $\mathcal{C}_p$ spaces seems to have grown out of early work of von Neumann and Schatten of which [**336**] is a good example and Schatten expanded on this in [**334, 335**]. In fact, according to McCarthy [**280**], it may have been a paper of von Neumann [**374**] which was the real beginning of the subject, as well as the beginning of the study of the non-commutative L^p spaces considered in Section 4. Other good references for study of the $\mathcal{C}_p$ spaces include [**146**] and [**319**].

Theorem 11.2.2 and its proof are taken from the 1969 paper of Russo [**331**], who seems to be the first person to actually describe the surjective isometries of a $\mathcal{C}_p$ space. Arazy took up the whole project in his Ph.D. thesis and published his work in [**15**]. The main theorem is as stated in 11.2.3. We note that our tendency has been to consider the cases for $1 \leq p$ even though the result holds also in the non Banach spaces when $0 < p < 1$. Arazy followed two years later with a joint article with Friedman [**17**] in which they described the nonsurjective isometries as well for $1 \leq p, p \neq 2$ and also gave a condition on a subspace of $\mathcal{C}_p$ in order that it be the range of a contractive projection.

The proof we have given for Arazy's theorem is due to Erdos [**118**]. Theorem 11.2.6 is due to McCarthy and the proof appears in [**280**].

The key result for the isometries of the Schatten classes is to show that such operators must preserve the rank for finite rank operators. Several papers concerned with preservation of rank were mentioned in Chapter 10. The paper of Prugovecki and Tip [**305**] contains a lemma (Lemma 4.2) which gives the form of such operators. This is essentially the same as the first lemma of Erdos [**118**]. The paper of Prugovecki and Tip also appeared the same year as that of Arazy. They were interested in semi groups of rank-preserving operators on normed ideals and do not have any theorems on the isometries of these normed ideals. We note that in [**43**] and [**44**], Arazy's theorem was used to describe the strongly continuous one-parameter groups of isometries on $\mathcal{C}_p$ for $1 \leq p \leq \infty$.

As we mentioned in Chapter 10, the results in this section also hold for the finite-dimensional case and so cover the description of isometries for the (p, k) norms when $k = m = n$ and $p \neq 2$. In a very interesting paper [**90**], Chan, Li, and Tu have extended the notion of s-numbers to bounded operators on an infinite- dimensional Hilbert space. This enables them to study isometries on a class of unitarily invariant norms that generalize the finite-dimensional (p, k) norms.

Isometries of Symmetric Norm ideals: Sourour's Theorem. The results in this section come from the 1981 paper of Sourour [**355**] and they provide an alternate proof to the result on the isometries of $\mathcal{C}_p$ treated in the previous sections. Lemma 11.3.1 is given in [**335**, p. 71], and the result used in the proof and attributed to Fischer-Courant is discussed in some detail by Schatten on pages 21-22 of [**335**]. The other lemmas and corollaries in the section are taken from Sourour's paper cited above. We did add our own proof of Corollary 11.3.3. We have followed Sourour in using the duality determined by $\langle\langle T, S\rangle\rangle = tr(S^*T)$ where S is of finite rank. This differs from the duality used in the earlier section.

We wish to state here two other results of Sourour in [**355**]. Let $\mathcal{G}$ denote the group of surjective isometries on $(\mathcal{J}, \nu)$ and let $\mathcal{G}_0$ denote the set of isometries of the form $e^{i\Psi}$ where Ψ is a Hermitian operator. Then $\mathcal{G}_0$ consists of the isometries of the form $\Phi(T) = UTV$ and $\mathcal{G}_0$ is a subgroup of $\mathcal{G}$.

11.5.1. Proposition. *(Sourour) The group $\mathcal{G}$ contains precisely two components, and the component containing the identity is $\mathcal{G}_0$. The subgroup $\mathcal{G}_0$ is normal and $\mathcal{G}/\mathcal{G}_0$ is the discrete group of two elements.*

We noted the same result about a normal subspace for admissible sequence spaces in Chapter 9, Theorem 9.2.13. In fact, it is true that on any Banach space X, if the set of exponentials of Hermitian operators forms a group, it is a normal subgroup of the group of isometries.

11.5.2. Proposition. *(Sourour) Let $(\mathcal{J}, \nu)$ be a minimal norm ideal different from $\mathcal{C}_2$. Then $\{\Phi_t : t \in \mathbb{R}\}$ is a strongly continuous group of isometries on $\mathcal{J}$ if and only if there exist self-adjoint (not necessarily bounded) operators A and B on $\mathcal{H}$ such that*

$$\Phi_t(T) = e^{itA}Te^{itB}.$$

If α is the infinitesimal generator of $\{\Phi_t\}$, then

$$\alpha(T) = i(AT + TB)$$

for all T in the domain of α. The domain of α is precisely the set of all operators T in $\mathcal{J}$ which map the domain of B into the domain of A and for which the closure of $AT + TB$ belongs to $\mathcal{J}$. The group $\{\Phi_t\}$ is uniformly continuous if and only if A and B are bounded.

This result generalizes a similar one for $\mathcal{C}_p$ found in [**43**], and the proof is the same. There are results for operators on Banach spaces rather than Hilbert spaces. One of the earlier results along these lines is due to Grzaslewicz [**168**] on isometries of $\mathcal{L}(\ell^p, \ell^r)$. We mention also the paper of Khalil and Saleh [**219**] and the survey paper of Rao on generalizations of Kadison's theorem [**315**]. The interested reader should consult these papers and their reference lists.

Noncommutative L^p Spaces. The study of noncommutative integration spaces has surprisingly early roots. Perhaps the earliest formal treatment is that of Segal [**339**] in 1953, which according to him was inspired by papers of von Neumann and Murray.

In this paper just cited, Segal develops a notion of measurability for operators and introduces the noncommutative L^1 and L^2 spaces. In his doctoral dissertation directed by Segal, Kunze [**227**] defines the L^p spaces for the other values of p. Nelson [**291**] gives an account of noncommutative integration theory, which he says uses no deep properties of von Neumann algebras. (See also [**392**].) There are many references to work of Dixmier and others which we do not list here. Our little description of how to view a classical L^p space as the L^p space of a commutative von Neumann algebra is taken from Segal's paper. We also cannot ignore the influence of Kadison's work [**209**], which is so prominent in the noncommutative case. As we mentioned earlier, the earliest treatment, although for finite-dimensional spaces, was probably the paper of von Neumann [**374**].

The definition we have given for $L^p(\mathcal{M}, \tau)$ is the one usually found today, and follows the discussion given in [**302**]. Russo [**330**] was interested early in studying the isometries on these spaces and it is his proof which we have given for Theorem 11.4.2. A number of the properties concerning duality between the $L^p(\mathcal{M}, \tau)$ spaces he attributes to Dixmier [**113**]. The proof of Theorem 11.4.4 is essentially an exact reproduction of the proof given by Yeadon in [**393**]. The omitted proof of Theorem 11.4.3 can also be found there. Yeadon's work extended the knowledge of (not necessarily surjective) isometries to the case of semifinite traces. Other early articles concerning isometries in the noncommutative case include [**55**], [**215**], and [**366**].

More recently, Sherman (and co-authors) has established a number of results going beyond the setting of semifinite von Neumann algebras [**208**], [**345**], and [**346**]. These papers are full of interesting discussion and insight as well as new results and proofs in general settings. The reference lists are invaluable sources of contributors to the general problem and we encourage the interested reader to consult them. We should mention the name of Watanabe, who published several papers in the 1990s on isometries on the noncommutative L^p spaces. We content ourselves with two references, [**380**] and [**381**].

We close by citing another paper of Sherman [**344**] in which he combines an observation of Paterson and Sinclair [**298**] and a variation of the methods of Yeadon [**393**] and himself [**345**] to give a new proof of the (Kadison) theorem on the structure of surjective isometries between (nonunital) C^*-algebras. Sherman ends this paper with a brief discussion of how one might adapt the method of proof to obtain a description of the surjective isometries of noncommutative L^p spaces $0 < p < \infty, p \neq 2$, which is new when $p < 1$.

We list the following papers as also related to the material in this chapter: [**1**], [**94**], [**299**], [**300**], [**351**], and [**360**].

CHAPTER 12

Minimal and Maximal Norms

12.1. Introduction

In this chapter we depart from the general theme of the volume, which has featured spaces of vector-valued functions. Here we are concerned with the size of the group of isometries on a given space.

If x, y are elements on the surface of the unit ball of a Hilbert space X, then it is well known that there is a surjective isometry T of the space such that $Tx = y$. The norm of a Banach space which has this property is said to be *transitive*. An equivalent way to describe transitivity of the norm is to say that for each $x \in X$ with $\|x\| = 1$, the orbit $\mathcal{G}(x) = \{Tx : T \in \mathcal{G}\}$ is equal to the surface $S(X)$ of the unit ball of X, where, as usual, $\mathcal{G} = \mathcal{G}(X)$ denotes the group of surjective isometries on $(X, \|\cdot\|)$. A question that has remained open since the time of Banach asks whether a separable Banach space with a transitive norm must be a Hilbert space.

Associated with transitivity are some weaker notions which have been considered through the years. Thus, a norm is *almost transitive* if $\mathcal{G}(x)$ is dense in $S(X)$ for each $x \in S(X)$, and the norm is *convex transitive* if the convex hull of $\mathcal{G}(x)$ is dense in $B(X)$ for each x of norm one in X. Finally we say that the norm on X is *maximal* if no equivalent norm on X has a strictly larger group of isometries. Clearly transitivity implies almost transitivity, which in turn implies convex transitivity. We will show later that a convex transitive norm must be maximal.

On the other hand, for any Banach space X, the group $\mathcal{G}$ must contain the subgroup consisting of all isometries of the form tI, where t is a scalar with $|t| = 1$. Isometries of this type are called *trivial isometries* and a norm for which the isometry group consists only of trivial isometries is called a *minimal norm*. In Section 2, we present Davis' construction of an infinite-dimensional real Banach space for which the only isometries (surjective or not) are $\pm I$. In Section 3, we give some results on spaces with trivial isometries, including Jarosz's proof that on any Banach space there is an equivalent norm which is minimal.

In Section 4, we return to the study of maximal norms. We examine some of the known implications among the various forms of transitivity, and some of the spaces for which the norm is maximal.

12.2. An Infinite-Dimensional Space with Trivial Isometries

This short section is devoted to the construction of the space advertised in its title. The space X involved will be the direct sum of ℓ^2 with $\mathbb{R}$ and we will let e_j denote the jth element of the standard orthonormal basis for ℓ^2. By e_0 we will mean the unit vector in $\mathbb{R}$. For each positive integer n let $\Lambda_n = \left\{te_n : -\frac{1}{2n+1} \le t \le \frac{1}{2n}\right\}$, define F in $\ell^2 \oplus \{0\}$ by $F = \overline{co}\{\Lambda_n : n = 1, 2, \dots\}$. If B_2 denotes the unit ball in ℓ^2, let B be the convex hull of the sets $\{F + e_0, -F - e_0, B_2\}$. The norm on our space X is then defined by requiring B to be the unit ball. We list some pertinent facts in the following lemma.

12.2.1. Lemma.

(i) *The only line segment of length* $1/3 + 1/2$ *in* F *is* Λ_1*; the only line segment of length* $1/(2n+1) + 1/2n$ *in* $F \cap \overline{sp}\{\Lambda_1, \dots, \Lambda_{n-1}\}^\perp$ *is* Λ_n*.*

(ii) *The extreme points of* B *are given by*

$$S_2 \oplus \{0\} \cup \left\{\pm\left(\frac{e_n}{2n} + e_0\right)\right\}_{n=1}^{\infty} \cup \left\{\pm\left(\frac{e_n}{2n+1} - e_0\right)\right\}_{n=1}^{\infty},$$

where S_2 *denotes the unit ball of* ℓ^2*.*

(iii) $B \subset \overline{co}\{2e_0, -2e_0, B_2\}$.

(iv) *If* Y *is a subspace of* X *with dimension at least* 3*, and if* $x \in Y \cap S_2 \oplus \{0\}$*, then* x *is a nonisolated extreme point of the unit ball of* Y*.*

(v) *If* T *is an isometry of* X *into itself, then the line segment* $T(\Lambda_1 + e_0)$ *does not meet* $S_2 \oplus \{0\}$*.*

(vi) *If* Y *is a two-dimensional subspace of* X*, and* Y *is isometric with* $\ell^2(2)$*, then* Y *is contained in* $\ell^2 \oplus \{0\}$*.*

Proof. We provide only a sketch. The first two items are clear from the construction of the space. If A denotes the set described in (iii), then the Minkowski functional of A is given by $p(u + ae_0) = \frac{1}{2}|a| + \|u\|$, where $u \in \ell^2 \oplus \{0\}$. To establish (iii), it is enough to show that $F + e_0$ is contained in A and this follows by checking the extreme points of $F + e_0$.

Suppose that Y and x are as given in (iv) above. Then $Y = H \oplus sp\{f_0\}$ where $H \subset \ell^2$ with dimension greater than or equal to 2, or else $Y = H \subset \ell^2 \oplus \{0\}$. Now x is an extreme point of B, and so of the ball in H as well. Since the dimension of H is at least 2, any neighborhood of x contains points of $H \cap S_2 \oplus \{0\}$ which are also extreme points of B.

If T is an isometry on X, then every point of $T(X) \cap S_2 \oplus \{0\}$ is extreme in the ball of X and so also in the ball of $T(X)$. Hence, $T(\Lambda_1 + e_0)$ can not intersect $S_2 \oplus \{0\}$ in an internal point of the segment. The end points of $\Lambda_1 + e_0$ are isolated extreme points of $T(X)$ and so cannot be in $S_2 \oplus \{0\}$ by (iv). This gives (v).

Finally, suppose that Y is a two-dimensional subspace of X which is isometric to $\ell^2(2)$. Then Y is contained in some three-dimensional subspace Z spanned by e_0, x_1, x_2, where x_1, x_2 are orthogonal unit vectors in $\ell^2 \oplus \{0\}$. Suppose that $Y = sp\{\alpha e_0 + \beta x_1, x_2\}$. It follows from (iii) that the ball of Z is supported at x_2 by the planes $\{ae_0 + bx_2 + cx_1 : \frac{1}{2}a + b = 1\}$ and $\{ae_0 + bx_2 + cx_1 : b - \frac{1}{2}a = 1\}$. Therefore, the unit ball of Y is supported at x_2 by the lines

$$\left\{\lambda\alpha e_0 + \lambda\beta x_1 + \mu x_2 : \frac{1}{2}\lambda\alpha + \mu = 1\right\}$$

and

$$\left\{\lambda\alpha e_0 + \lambda\beta x_1 + \mu x_2 : -\frac{1}{2}\lambda\alpha + \mu = 1\right\}.$$

However, since Y is isometric to $\ell^2(2)$, there must be a unique supporting line at x_2, and this can only happen if $\alpha = 0$. Hence, $Y = sp\{x_1, x_2\} \subset \ell^2 \oplus \{0\}$. □

We observe that if T is an isometry from X into itself, and W is any two-dimensional subspace of $\ell^2 \oplus \{0\}$, then $T(W) \subset \ell^2 \oplus \{0\}$ by (vi) above. Hence, $T(\ell^2 \oplus \{0\}) \subset \ell^2 \oplus \{0\}$.

12.2.2. THEOREM. *If T is an isometry from X into X, where X is the space described above, then $T = \pm I$.*

PROOF. If T is the prescribed isometry, then $T(\Lambda_1 + e_0)$ must lie entirely above $\ell^2 \oplus \{0\}$ or entirely below by Lemma 12.2.1 (v). We will assume it lies above. By part (vi) of the lemma, we have $Te_1 = f_1 \in \ell^2 \oplus \{0\}$, and so $T(\Lambda_1 + e_0) = \Lambda_1' + f_0$ where $\Lambda_1' = \{tf_1 : -\frac{1}{3} \le t \le \frac{1}{2}\}$. Since this segment is above $\ell^2 \oplus \{0\}$, we have that $f_0 = ae_0 + x$ with $a > 0$ and $x \in \ell^2 \oplus \{0\}$. Because T is an isometry, the endpoints of $\Lambda_1' + f_0$ lie in S, the surface of the unit ball of X. It follows that there exist vectors $u, v \in F + e_0$ and unit vectors y, z in $\ell^2 \oplus \{0\}$ such that $f_0 + \frac{1}{2}f_1 = \lambda y + (1-\lambda)u$ and $f_0 - \frac{1}{3}f_1 = \mu z + (1-\mu)v$, where $\lambda, \mu \in [0,1]$. Let us suppose that $f_0 + \Lambda_1'$ lies strictly between $\ell^2 \oplus \{0\}$ and $F + e_0$. This means that $0 < a, \lambda, \mu < 1$, and it can be shown that $a = (1-\lambda) = (1-\mu)$. If $p = \alpha(f_0 - \frac{1}{3}f_1) + (1-\alpha)(f_0 + \frac{1}{2}f_1)$ is an internal point of $\Lambda_1' + f_0$, we have

$$\|p\| \le (1-a)\|\alpha z + (1-\alpha)y\| + a\|\alpha u + (1-\alpha)v\|.$$

Since $\|y\| = \|z\| = 1$, the strict convexity of $\ell^2 \oplus \{0\}$ implies that $\|\alpha z + (1-\alpha)y\| < 1$ if $y \ne z$. But this would imply that $\|p\| < 1$, which is not the case. Hence, $y = z$. It can be seen from this that the line segment $\Lambda_1' + f_0$ is parallel to the base of the triangle with vertices $y, u,$ and v. It's length, therefore, must be less than the length of the segment connecting u and v, which lies in $\Lambda_1' + f_0$ and whose length is no more than $1/2 + 1/3$ by (i) of the lemma. This is impossible because T preserves the length of segments, and the conclusion is that $f_0 + \Lambda_1'$ is a segment of length $1/2 + 1/3$ lying in $F + e_0$. Therefore, it is the segment $\Lambda_1 + e_0$, and so $f_0 = e_0$ or $f_0 = e_0 + \frac{1}{6}e_1$ depending on how T maps the end points. Hence $Te_1 = e_1$ or $-e_1$ (respectively). Similarly, $T(\Lambda_2 + e_0)$ is also parallel to $\ell^2 \oplus \{0\}$, orthogonal to $\Lambda_1' + f_0$, and has length $1/4 + 1/5$. We conclude by Lemma 12.2.1 (ii) that $T(\Lambda_2 + e_0) = \Lambda_2 + e_0$.

From this we have $e_0 = f_0, Te_1 = e_1$, and $Te_2 = e_2$. Proceeding by induction, we can show that $Te_n = e_n$ for each n and so $T = I$. The same argument would give $T = -I$ if we assumed that $T(\Lambda_1 + e_0)$ lies below $\ell^2 \oplus \{0\}$. $\square$

We remark that the proof of the previous result would be much easier had we assumed that T is surjective. In that case, T would have to map extreme points to extreme points, and so the preservation of the segments $\Lambda_n + e_0$ by T would follow quickly.

12.3. Minimal Norms

In the previous section we have shown that there exist Banach spaces with minimal norms; that is, spaces for which the isometry group consists of modulus 1 multiples of the identity. We want to show now that every Banach space has an equivalent minimal norm. First we establish a lemma that shows the above statement is true for certain subspaces of $\ell^\infty(\Gamma)$.

12.3.1. LEMMA. *(Jarosz) Let Γ be a set and E a Banach space such that $c_0(\Gamma) \subset E \subset \ell^\infty(\Gamma)$. Then there exists a norm ν on E which is equivalent to the usual sup norm $\|\cdot\|_\infty$ of E and for which a linear map T on E is an isometry for both ν and $\|\cdot\|_\infty$ if and only if T is a modulus 1 multiple of the identity I on E.*

PROOF. Suppose T is an isometry of E with respect to $\|\cdot\|_\infty$. A bit of calculation shows that we can conclude that a pair of norm 1 functions e', e'' do not have disjoint supports if and only if there exists $w \in B(E)$ and modulus 1 scalars α, β such that

$$\|e' + \alpha e'' + \beta w\|_\infty > 1$$

and

$$\|e' + \lambda w\|_\infty \leq 1, \quad \|e'' + \lambda w\|_\infty \leq 1 \text{ for all } |\lambda| \leq 1.$$

For, suppose e', e'' are elements of E with norm 1 and for some $t \in \Gamma$ we have $e'(t) = re^{i\theta}, e''(t) = se^{i\varphi}$, where $r, s \neq 0$. Let $w = qe_t$ where $1 - (r+s) < q < min(1-r, 1-s)$ if $r + s < 1$ and $q = 0$ otherwise. (Here, as usual, e_t denotes the characteristic function of $\{t\}$.) Let $\alpha = e^{i(\theta-\varphi)}$ and $\beta = e^{i\theta}$.

As an isometry, T must preserve these linear and metric properties, and it follows from the above conditions that T must map elements with disjoint supports to elements with disjoint supports. By well-known arguments (see, for example, pp. 49-50 of Chapter 3), we may conclude that

$$Te_t = h(t)e_{\pi(t)} \text{ for all } t \in \Gamma$$

where π is a permutation of Γ and $|h(t)| = 1$ for all $t \in \Gamma$.

Let $<$ denote a fixed well ordering of Γ and define ν on E by

$$\nu(x) = \max\{\|x\|_\infty, \sup\{|2x(t) + x(s)| : t < s \in \Gamma\}\}.$$

Assume that T is also an isometry for ν. To complete the proof we must show that π is the identity permutation and that $h(t) = h(s)$ for $t, s \in \Gamma$. To show

π is the identity, it is sufficient to show it preserves the order on Γ. Hence, let us suppose that $t < s$ but $\pi(t) > \pi(s)$. Then $\nu(2e_t + e_s) = 5$ but

$$\nu(T(2e_t + e_s)) = \nu(2h(t)e_{\pi(t)} + h(s)e_{\pi(s)}) = \max\{2, |2h(t) + 2h(s)|\} \leq 4,$$

which is a contradiction. If we assume that $h(t) \neq h(s)$, then $\nu(e_t + e_s) = 3$ while

$$\nu\left(T(e_t + e_s)\right) = \nu\left(h(t)e_t + h(s)e_s\right) = \max\{1, |2e_t + e_s|\} < 3.$$

□

12.3.2. Lemma. *(Jarosz) Let $(X, \|\cdot\|)$ be a Banach space, x_0 a nonzero element of X, $p(\cdot)$ a continuous norm on $(X, \|\cdot\|)$, G_1 the group of isometries on $(X, \|\cdot\|)$, and G_2 the group of isometries T of (X, p) such that Tx_0 and x_0 are linearly dependent. Then there is a norm ν on $Y = X \oplus \mathbb{F}$ which coincides with $\|\cdot\|$ on X and such that the group G of all isometries of (Y, ν) is isomorphic to $G_1 \cap G_2$.*

Proof. First note that we may assume that p is equivalent with the norm on X. If not, we let

$$p'(x) = \|x\| + p(x)$$

and observe that p' is a continuous norm that is equivalent to the original norm and that a linear operator T on X preserves both $\|\cdot\|$ and p if and only if it preserves both $\|\cdot\|$ and p'. We may also assume (through multiplication by an appropriate positive number if necessary) that

$$1000\|x\| \leq p(x) \ \text{ for } x \in X \ \text{ and that } \ \|x_0\| \leq 0.1.$$

Let

$$A = \{(x, \alpha) \in X \oplus \mathbb{F} : \max\{\|x\|, |\alpha|\} \leq 1\},$$

$$C = \{(x + x_0, 2) : p(x) \leq 1\}.$$

Let ν denote the norm determined by the unit ball W which is the closed, balanced, convex hull of $A \cup C$. We see that

$$\nu((x, \alpha)) = \|x\| \ \text{ for all } \ (x, \alpha) \in Y, \ \ |\alpha| \leq \|x\|.$$

Thus, the norm ν coincides with $\|\cdot\|$ on X. Furthermore, if T is an operator on X which preserves both norms $\|\cdot\|$ and p, and $Tx_0 = \lambda x_0$ for $|\lambda| = 1$, then $T \oplus \lambda I_{\mathbb{F}}$ is an isometry of (Y, ν).

Suppose now that $\tilde{T}$ is an isometry on Y for the norm ν. The proof of the lemma is completed if it can be shown that there exists λ with $|\lambda| = 1$ such that

(i) $\tilde{T}$ maps $X = X \oplus \{0\}$ onto X;
(ii) $T = \tilde{T}_{|X}$ preserves both $\|\cdot\|$ and $p(\cdot)$;
(iii) $\tilde{T}(x_0, 0) = (\lambda x_0, 0)$ and $\tilde{T}((0, 1)) = (0, \lambda)$, where $|\lambda| = 1$.

Observe that λC is a face of W for each $|\lambda| = 1$, and that the points on the boundary of W that are not in the relative interiors of the faces λC must be points interior to a segment contained in the boundary of W whose length (with respect to the norm ν) is at least 0.1 or limits of such points. The isometry $\tilde{T}$ must preserve the types of points and must map a face of W onto a face of W. It follows that $\tilde{T}(C) = \lambda C$ for some $|\lambda| = 1$. By replacing $\tilde{T}$ by $\bar{\lambda}\tilde{T}$ if necessary, we may assume that $\tilde{T}C = C$. In particular, since $(x_0, 2) \in C$, we must have $\tilde{T}(x_0, 2) \in C$.

Let $x \in X$ with $p(x) \leq 1$. Then

$$\tilde{T}(x, 0) = \tilde{T}((x + x_0, 2) - (x_0, 2)) = \tilde{T}(x + x_0, 2) - T(x_0, 2) \in C - C \subset X \oplus \{0\},$$

and it follows that $\tilde{T}X \oplus \{0\} \subset X \oplus \{0\}$. By applying the same argument to $\tilde{T}^{-1}$, we have $\tilde{T}(X \oplus \{0\}) = X \oplus \{0\}$. Since ν agrees with $\|\cdot\|$ on X, it is clear that $T = \tilde{T}_{|X}$ is an isometry on $(X, \|\cdot\|)$.

If $(x, 2) \in C$, then $\tilde{T}(x, 2) = \tilde{T}(x, 0) + \tilde{T}(0, 2) \in C$, and since $\tilde{T}(x, 0) = (Tx, 0)$, it must be the case that $\tilde{T}(0, 2) = (u, 2) = (Tw, 2)$ for some $w \in X$. If we suppose that $x \in B = \{x : p(x - x_0) \leq 1\}$, then $(x, 2) \in C$ so that $Tx + Tw \in B$ as well. Now if $p(x) \leq 1$, we have

$$x_0 \pm x \in B \Rightarrow Tx_0 + Tw \pm Tx \in B$$

from which we obtain

$$\begin{aligned} p(Tx) &\leq \frac{1}{2}[p(Tx + (Tx_0 + Tw - x_0)) + p(Tx - (Tx_0 + Tw - x_0))] \\ &\leq 1. \end{aligned}$$

It is clear from this that we get $p(Tx) = p(x)$, so that (i) and (ii) above are established. Now we want to show that $Tx_0 = x_0$, and that the w defined above is actually 0. To this end, we observe that since $\tilde{T}(x_0, 2) = (Tx_0 + Tw, 2) \in C$, we must have $p(T(x_0 + w - q)) \leq 1$, where $Tq = x_0$. This implies that $((x_0 + w - q) + x_0, 2) \in C$, and if we apply $\tilde{T}$ to it, we conclude that

$$p(2Tx_0 + 2Tw - 2Tq) \leq 1,$$

from which it follows that

$$p(x_0 + w - q) \leq \frac{1}{2}.$$

We can continue in this way, by induction, to conclude that

$$p(x_0 + w - q) \leq \frac{1}{n} \text{ for all positive integers } n.$$

Hence, $q = x_0 + w$ or $Tx_0 = x_0 - Tw$. We claim that $w = 0$. Suppose not. Then $\nu\left(\frac{w}{\|w\|}, 1\right) = 1$ while

$$\tilde{T}\left(\frac{w}{\|w\|}, 1\right) = \left(T\left(\frac{w}{\|w\|}\right) + \frac{Tw}{2}, 1\right).$$

However,

$$\left\| T\left(\frac{t}{\|w\|}\right) + \frac{Tw}{2} \right\| = \left| 1 + \frac{\|w\|}{2} \right| > 1.$$

It follows that

$$\nu\left(\tilde{T}\left(\frac{w}{\|w\|}, 1\right)\right) > 1,$$

which contradicts the fact that $\tilde{T}$ is an isometry. Hence, we have $Tx_0 = x_0$ and $\tilde{T}(0,1) = (0,1)$. If $TC = \lambda C$ for $|\lambda| = 1$, we get (iii). □

12.3.3. THEOREM. *(Jarosz) For any Banach space X, there is a Banach space Y with $X \subset Y$ and $\dim Y/X = 1$ such that Y has only trivial isometries.*

PROOF. As a consequence of a result of Pličko [**303**] concerning existence of a bounded, total biorthogonal system in a Banach space, there exists a set Γ and a bounded linear injective map J from X into $\ell^\infty(\Gamma)$ such that the closure E of $J(X)$ contains $c_0(\Gamma)$. By Lemma 12.3.1, there is a norm on E for which the isometries are trivial. Fix $t \in \Gamma$, and consider

$$E \cong \{e \in E : e(t) = 0\} \oplus_\infty \mathbb{F}.$$

By Lemma 12.3.2, there is a continuous norm $\tilde{p}$ on E such that $(E, \tilde{p})$ has only trivial isometries. Now define a continuous norm on X by

$$p(x) = \tilde{p}(Jx), \quad x \in X.$$

It follows that $(X, p(\cdot))$ has only trivial isometries, and by Lemma 12.3.2 again, there is a norm on $Y = X \oplus \mathbb{F}$ with trivial isometries and which coincides with the original norm on X. □

Finally, we are able to state the main theorem.

12.3.4. THEOREM. *(Jarosz) For any Banach space X there is an equivalent norm on X which is minimal.*

PROOF. Let Z be a subspace of X such that X/Z has dimension 1. By Theorem 12.3.3, there is a Banach space $Y = Z \oplus \mathbb{F}$ with a norm that agrees with the norm on Z and for which all the isometries are trivial. It is straightforward to show that the norm on Y is equivalent to the norm on $X = Z \oplus \mathbb{F}$ since the norms of both X and Y agree on Z. □

One might ask whether a space could have a minimal norm which is also maximal. In the real case, this cannot happen.

12.3.5. THEOREM. *(Wood) Let X be a real Banach space with dimension greater than 1. Then there is an equivalent norm on X for which there is a nontrivial isometry.*

PROOF. Let $(X, \|\cdot\|)$ be as given in the statement and suppose x_0 is an element of the unit sphere. Let $f \in X^*$ with $1 = \|f\| = f(x_0)$ and define T by $Ty = y - 2f(y)x_0$. Then T is not a multiple of the identity, $T^2 = I$, and T is an isometry for the equivalent norm ν defined by

$$\nu(x) = max\{\|x\|, \|Tx\|\}.$$

□

12.4. Maximal Norms and Forms of Transitivity

In this section we wish to examine a bit more the string of implications indicating the various weaker forms of transitivity for the norm. Although the transitivity conditions are really properties of the norm, we will sometimes attribute them to the space X itself, meaning that the given or natural norm on X has the property.

As we mentioned earlier, we have that transitivity ⇒ almost transitivity (AT) ⇒ convex transitive (CT). Let us now show that a convex transitive norm must be maximal.

12.4.1. THEOREM. *(Rolewicz) Suppose X is a Banach space with a convex transitive norm. Then the norm on X is maximal, and any norm that is equivalent to the given norm must be a multiple of that norm.*

PROOF. Suppose $x \in X$ with $\|x\| = 1$. Since the norm is (CT), we know that $\overline{co}\{Tx : T \in \mathcal{G}(X)\} = B(X)$. Suppose that ν is an equivalent norm on X whose group $\mathcal{G}_1$ of surjective isometries contains $\mathcal{G}(X)$. Given $y \in S(X)$ and $\epsilon > 0$, there exist $U_1, \ldots, U_n$ in $\mathcal{G}$ and nonnegative real numbers $\lambda_1, \ldots, \lambda_n$ whose sum is one such that

$$\|y - \sum_{j=1}^{n} \lambda_j U_j(x)\| < \epsilon.$$

If we assume that b is a positive number such that

$$\nu(z) \leq b\|z\|$$

for all $z \in X$, then we have

$$\begin{aligned}
\nu(y) &\leq \nu\left(y - \sum \lambda_j U_j(x)\right) + \nu\left(\sum \lambda_j U_j(x)\right) \\
&\leq b\|y - \sum \lambda_j U_j(x)\| + \nu\left(\sum \lambda_j U_j(x)\right) \\
&\leq b\epsilon + \nu(x).
\end{aligned}$$

It follows that $\nu(y) \leq \nu(x)$ and the opposite inequality can be established in a similar way. Hence, we have that $S(X) \subset \{y \in X : \nu(y) = r\}$ for some $r > 0$. This shows that $\nu(x) = r\|x\|$ for all $x \in X$ so that the group $\mathcal{G}_1$ is the same as $\mathcal{G}$ and the norm is maximal. □

We note that the conclusion of the previous theorem is a bit stronger than maximality. In fact, the result is that no (essentially) different equivalent norm exists for which $\mathcal{G}$ is the isometry group. A norm with the property that there is no equivalent norm, not a linear multiple of the given norm, with the same group of isometries, is called *uniquely maximal.* It turns out that a uniquely maximal norm is convex transitive.

12.4.2. THEOREM. *(Cowie) The norm on a Banach space X is convex transitive if and only if it is uniquely maximal.*

PROOF. The necessity is given by Theorem 12.4.1. Suppose then that the given norm $\|\cdot\|$ is uniquely maximal but not convex transitive. Hence we may assume that there are norm 1 elements $y, z \in X$ such that z is not in the closed convex hull D of the $\mathcal{G}$-orbit of y. By the Hahn-Banach theorem, there exists $f \in X^*$ such that $|f(x)| \leq 1$ for all $x \in D$, but $|f(z)| > 1$. Let us define a norm ν on X by

$$\nu(x) = \|x\| + \sup\{|f(Ux)| : U \in \mathcal{G}\}.$$

Then ν is equivalent to $\|\cdot\|$ and for any $T \in \mathcal{G}$, we have

$$\begin{aligned}\nu(Tx) &= \|Tx\| + \sup\{|f(U(x)| : U \in \mathcal{G}\} \\ &= \|x\| + \sup\{|f(Ux)| : U \in \mathcal{G}\} \\ &= \nu(x)\end{aligned}$$

for all $x \in X$. This shows that the group $\mathcal{G}(\nu)$ of isometries for ν contains the group $\mathcal{G}$, and since $\|\cdot\|$ is maximal, they must be the same. However, the uniquely maximal property implies that there exists some $k > 1$ such that $\|x\| < \nu(x) = k\|x\|$ for all x. Therefore,

$$\begin{aligned}\sup\{|f(Ux)| : U \in \mathcal{G}\} &= \nu(x) - \|x\| \\ &= (k-1)\|x\| = r\|x\|,\end{aligned}$$

where $r > 0$. For any $x \in X$ and $U \in \mathcal{G}$, we have

$$|f(Ux)| \leq \|f\|\|Ux\| = \|f\|\|x\|.$$

Hence, $r \leq \|f\|$, and for any $\epsilon > 0$, there exists $x \in S(X)$ so that

$$\|f\| - \epsilon \leq |f(x)| \leq \sup\{|f(Ux)| : U \in \mathcal{G}\} = r.$$

We conclude that $\|f\| = r$ and for any $x \in X$,

$$\sup\{|f(Ux)| : U \in \mathcal{G}\} = r\|x\| = \|f\|\|x\|.$$

When we apply this to both z and y we have the contradiction

$$\begin{aligned}1 < |f(z)| \leq \|f\| &= \sup\{|f(Uz)| : U \in \mathcal{G}\} \\ &= \sup\{|f(Uy)| : U \in \mathcal{G}\} \\ &\leq 1.\end{aligned}$$

□

It is known that any finite-dimensional space with transitive norm must be an inner product space. Transitivity of the norm does seem to be a characteristic property of Hilbert spaces, although there exist noncomplete separable spaces with transitive norm that are not pre-Hilbert spaces. (An example is the linear subset of $L^p[0,1]$ of all functions which vanish close to the point 1.) Also, it is possible to construct a class of measurable sets Σ, and a measure μ on the set Ω formed as a product of $[0,1]$ with an uncountable set so that the nonseparable space $L^p(\Omega, \Sigma, \mu)$ has a transitive norm. However, there is no known separable Banach space with transitive norm other than Hilbert space. The question mentioned in the introduction to this chapter, sometimes called the Banach-Mazur problem, as to whether such a space is a Hilbert space, remains open.

There is a collection of classical spaces that satisfy the almost transitive property. In keeping with the spirit of this volume, we will show this to be true in the vector-valued case.

12.4.3. THEOREM. *(Rolewicz), (Greim, Jamison, and Kaminska) Let X be a Banach space with almost transitive norm, and suppose that $1 \le p < \infty$. Then $L^p([0,1], X)$ has almost transitive norm.*

PROOF. Let $F, G \in L^p([0,1], X)$ with $\|F\| = \|G\| = 1$ and let $\epsilon > 0$ be given. If $H \in L^p([0,1], X)$, we let $|H|$ denote the real-valued element of $L^p[0,1]$ defined by $|H|(t) = \|H(t)\|$. If $h \in L^p$ is nonvanishing, we can define T_h on L^p by

$$T_h(f)(t) = f(k_h(t))h(t),$$

where $k_h(t) = \displaystyle\int_0^t |h(s)|^p ds$. Then T_h is an isometry with inverse given by

$$T_h^{-1}(f)(t) = f(k_h^{-1}(t))\frac{1}{h(k_h^{-1}(t))}.$$

Observe that $T_h(1) = h$, so if we assume that F and G are nowhere vanishing, the operator U defined by

$$U = T_{|G|}T_{|F|}^{-1}$$

is an isometry on L^p with the property that $U|F| = |G|$. By Lamperti's theorem (3.2.5) we can describe U by the form

$$Uf(t) = h(t)\tau f(t),$$

where τ is the isometry induced by a regular set isomorphism, and h satisfies the properties of that theorem. We can define, for $x \in X$ and measurable set E, $U\chi_E(t)x = h(t)\chi_{\tau E}(t)x$ and extend this to all simple functions. It follows that $|UH| = U|H|$ for all simple functions and, by density of simple functions and continuity, for all $H \in L^p([0,1], X)$. If we assume that F and G are nonvanishing simple functions, we have $|UF| = U|F| = |G|$, and we can write

$$UF(t) = \sum_{j=1}^{n} x_j \chi_{A_j}(t), \quad G(t) = \sum_{j=1}^{n} y_j \chi_{A_j}(t),$$

with $\|x_j\| = \|y_j\| \neq 0$, and the A_j's a disjoint partition of $[0,1]$. Since the norm of X is AT, we can find isometries U_j of X such that $\|U_j x_j - y_j\| < \epsilon\|x_j\|$ for each $j = 1, \ldots, n$. For each t let $U_t = \sum_{j=1}^{n} U_j \chi_{A_j}(t)$. If $VH(t) = U_t UH(t)$ for each t, then V is an isometry which satisfies

$$\|VF - G\|^p = \sum_{j=1}^{n} \|U_j x_j - y_j\|^p \lambda(A_j) \leq \sum_{j=1}^{n} \epsilon^p \lambda(A_j) = \epsilon^p.$$

Because the nonvanishing simple functions are dense in $L^p([0,1], X)$, we conclude that the norm is AT.

□

We see, then, that the classical L^p spaces have almost transitive norms. However, their cousins, the ℓ^p spaces (for $p \neq 2$) do not. For suppose $x = e_1$, $y = e_1 + e_2$, and $z = y/\|y\|$, where e_k denotes an element of the usual unit vector basis of ℓ^p. If U is any isometry of ℓ^p, then $Ux = ae_k$ for some scalar a with $|a| = 1$. It is easily seen that $\|Ux - z\| \geq 1/\|y\|$, so that the orbit of x under the isometry group cannot be dense in the unit sphere. The uniform norm on the $C(K)$ spaces is also never AT (where K is compact and not a singleton). This is clear, since the orbit of an extreme point of the unit ball must lie within the extreme points and so cannot be dense in the unit sphere.

In fact, the norms of these spaces are not convex transitive either. For example, consider the complex space $C[0,1]$ with the norm ν defined by

$$\nu(f) = \|f\|_\infty + |f(0)| + |f(1)|.$$

Then ν is clearly equivalent to the uniform norm $\|\cdot\|_\infty$ on $C[0,1]$ and the two norms have the same group of isometries. From this we see that the uniform norm is not uniquely maximal, and so not convex transitive. It is maximal, however, and we will show that shortly. It is known that the complex spaces $L^\infty[0,1]$, $C(K)$, where K is the Cantor set, and $C(\mathbb{T})$, where $\mathbb{T}$ is the unit circle, are convex transitive (that is, their natural norms are CT), but not AT. We note here that the Hardy spaces H^p on the disk, for $1 \leq p < \infty, p \neq 2$, are not almost transitive. (See the first section of Chapter 13 for an argument.)

Let us turn our attention now to the problem of showing that the norm on certain $C(K)$ spaces is maximal. Suppose K is a locally compact Hausdorff space. As a consequence of the Banach-Stone theorem (Corollary 2.3.12), the group $\mathcal{G}$ of surjective isometries for $C_0(K)$ can be thought of as a product of the group of multipliers M_h and the group Γ induced by the homeomorphisms on K. By a multiplier M_h, we mean the operator defined by $M_h f = hf$, where h is continuous and of modulus 1. We need to recall also the notion of Hermitian element from Chapter 9. The following lemma is crucial.

12.4.4. LEMMA. *(Kalton and Wood) Let $\|\cdot\|$ be a norm on the complex space $C_0(K)$ equivalent to the uniform norm such that every multiplier M_h is an isometry. Then there exists an equivalence relation $\sim$ on K such that for some positive integer n, the cardinality of each equivalence class is no more than n, and x^* is a Hermitian element of $C_0(K)^*$ if and only if*

$$x^* = \sum_{s \sim t} a_s \psi_s$$

for some $t \in K$, where $a_s \in \mathbb{C}$ and ψ_s is the unit mass measure (or evaluation functional) at s.

We omit the proof of this lemma, but we note that the equivalence relation mentioned in the statement is defined by $t \sim t'$ if and only if ψ_t and $\psi_{t'}$ belong to the same Hilbert component of $C_0(K)^*$.

12.4.5. THEOREM. *(Kalton and Wood) Let K be a locally compact Hausdorff space such that either*

(i) *there is a dense subset K_0 of points possessing a neighborhood homeomorphic to an open set in Euclidean space; or*
(ii) *K is infinite and possesses a dense set of isolated points.*

Then $\mathcal{G}$ is a maximal bounded group so that the uniform norm on the complex space $C_0(K)$ is maximal.

PROOF. Let $\|\cdot\|$ be a norm on the complex space $C_0(K)$ equivalent to the uniform norm whose group of isometries contains $\mathcal{G}$. Let $\sim$ be the equivalence relation that comes from Lemma 12.4.4. If γ is a homeomorphism of K and $t \sim t'$, then T_γ given by $T_\gamma f(s) = f(\gamma(s))$ is an isometry. Since $T_\gamma^*(\psi_t + \psi_{t'}) \in h(C_0(K)^*)$, we obtain from the lemma the fact that $\gamma(t) \sim \gamma(t')$.

Let us assume that (i) holds, and suppose $t \in K$ has a Euclidean neighborhood W and $t \sim t'$ with $t \neq t'$. There is a closed neighborhood B of t such that B is contained in the interior of W, $t' \notin B$, and B is homeomorphic to a closed ball in $\mathbb{R}^n$ for some n. For any $t'' \in \operatorname{int} B$, there is a homeomorphism φ from B onto itself such that $\varphi(t) = t''$, and φ is the identity on the boundary of B. Now define γ by $\gamma(t) = \varphi(t)$ for $t \in B$ and $\gamma(t) = t$ for $t \notin B$. Then γ is a homeomorphism of K with $\gamma(t) = t''$ and $\gamma(t') = t'$. By the remark at the end of the previous paragraph, we conclude that $t'' \sim t'$, so that $t'' \sim t' \sim t$. It would follow then that the set $\{t' : t' \sim t\}$ has infinite cardinality, contradicting the lemma. Hence we must conclude that if $t \sim t'$, then $t = t'$.

Let K_0 denote the set of all $t \in K$ which have Euclidean neighborhoods, and suppose $U : C_0(K) \to C_0(K)$ is an isometry for the norm $\|\cdot\|$. For $t_0 \in K_0$, $U^*\psi_{t_0}$ belongs to a Hilbert component of $C_0(K)^*$ of dimension 1. Therefore, since $\{\psi_t : t \in K\}$ is a maximal orthonormal system, we must have

$$U^*\psi_{t_0} = f(t_0)\psi_{\gamma(t_0)},$$

where $|f(t_0)| = 1$ and γ is a map from K_0 to K. By the weak*-continuity of U^*, it follows that

$$U^*\psi_t = f(t)\psi_{\gamma(t)},$$

where f is continuous and of modulus 1 on K, and γ is continuous from K into K. Since U is invertible, we must conclude that γ is a homeomorphism and U is an isometry for the uniform norm.

If (ii) holds, the proof is essentially the same. If t is an isolated point and $t \sim t'$, we can, for any other isolated point t'', find a homeomorphism γ on K so that $\gamma(t) = t''$, $\gamma(t'') = t$, and $\gamma(s) = s$ for s different from both t and t''. Thus $t'' \sim t'$, and we again have that the equivalence class of t is infinite, a contradiction. The remainder of the proof is the same as in the previous paragraph.

□

In particular, we see that the uniform norm for the complex space $C[0,1]$ is maximal. The situation is different for real spaces.

12.4.6. Theorem. *(Partington) The uniform norm on the real space $C([0,1],\mathbb{R})$ is not maximal.*

Proof. Define the linear map T from $C([0,1],\mathbb{R})$ to itself by

$$Tf(t) = f(t) - f(0) - f(1).$$

It is easy to show that $T^2 f = f$, and if we let

$$\nu(f) = \max\{\|f\|, \|Tf\|\},$$

then ν is a norm which is equivalent to the original norm $\|\cdot\|$, and T is an isometry for this new norm.

Now suppose that U is an isometry for the original norm, so that $Uf(t) = hf(\varphi(t))$ for all $t \in [0,1]$, where φ is a homeomorphism of $[0,1]$, and h is either 1 or -1. Therefore,

$$\begin{aligned}\nu(Uf) &= \max(\|Uf\|, \|TUf\|) = \max\left(\|f\|, \|f\circ\varphi - f(\varphi(0)) - f(\varphi(1))\|\right)\\ &= \max\left(\|f\|, \|f\circ\varphi - f(0) - f(1)\|\right) = \nu(f).\end{aligned}$$

These equalities hold because φ must map end points of the interval to end points, and $\|Tf\| = \|T(f\circ\varphi)\|$. The conclusion is that U is an isometry for ν as well, so that the isometry group for ν is strictly larger than the group of isometries for the uniform norm. □

It is not known whether the norm on $C(\mathbb{T},\mathbb{R})$ is maximal, where $\mathbb{T}$ is the unit circle.

The results we have discussed are only a small part of those contained in an extensive literature on the subject. A bit more is said in the notes below, along with some guides for further reading.

12.5. Notes and Remarks

The question of whether a separable space with a transitive norm must be a Hilbert space is stated by Banach [**18**, p. 242], where he attributes the question to Mazur. The various notions of transitivity and the idea of a maximal norm were first put forth by Pelczynski and Rolewicz at the International Mathematical Congress at Stockholm in 1962, and published by Rolewicz in 1972 [**324**, Chapter IX]. (Our reference is to the second edition published in 1984.)

The first reference to trivial isometries seems to be the paper of Davis [**103**], although as Davis points out, the question of whether such a space exists was asked by I. Singer, and the question evidently involved not just surjective isometries, but all isometries. Also, Pelczynski had communicated to Davis an example of a compact Hausdorff space Q whose only homeomorphism is the identity, so that the only surjective isometries on $C(Q,\mathbb{R})$ would be $\pm I$, by virtue of the Banach-Stone theorem.

Trivial Isometries. The exposition in this section comes directly from the paper of Davis [**103**] mentioned above. It seems to be the first construction of a space with the express purpose of having only trivial isometries. Furthermore, this space constructed by Davis is the only one known to us in which the only isometries (surjective or not) are trivial. Other results of this type refer to surjective isometries.

At the suggestion of I. Singer, Davis was motivated to develop this construction in order to answer a question raised by him in a paper on positive bases in Banach spaces [**102**]. A (Schauder) basis $\{x_n\}$ for a Banach space X is said to be *positive* if whenever T is an isometry of X into X, and $Tx_n = \sum a_{nk}x_k$, there is a second isometry T^+ on X such that $T^+x_n = \sum |a_{nk}|x_k$. The question raised by Davis in [**102**] is this: If $\{x_n\}$ is a positive basis, is it unconditional? Since the space X constructed in this section has only $\pm I$ as isometries, every basis is positive. On the other hand, since X has a basis $\{e_k : k = 0, 1, 2, \dots\}$, it must also have a conditional basis [**301**]. Hence, the question has a negative answer.

It is natural to ask if there are previously known Banach spaces that have only trivial isometries. An obvious target for such an investigation might well be the space of James [**192**], and in the mid-1980s several authors did just that. In fact, when people refer to the James space, they may mean the space J equipped with any of several equivalent norms. The space of the paper referred to above is the space of zero-convergent sequences for which the following is finite:

$$\nu_1(x) = \sup \left[\sum_{j=1}^{n} |x_{p_j} - x_{p_{j+1}}|^2 + |x_{p_{n+1}} - x_{p_1}|^2 \right]^{1/2},$$

where the supremum is taken over all positive integers n and all finite increasing sequences of at least two positive integers $p_1, \dots, p_{n+1}$. Some other equivalent norms on J include

$$\nu_2(x) = \sup \left[\sum_{j=1}^{n} |x_{p_{2j-1}} - x_{p_{2j}}|^2 + |x_{p_{2n+1}}|^2 \right]^{1/2},$$

$$\nu_3(x) = \sup_{p_1 < \cdots < p_{2n}} \left[\sum_{j=1}^{n} |x_{p_{2j-1}} - x_{p_{2j}}|^2 \right]^{1/2},$$

$$\nu_4(x) = \frac{1}{\sqrt{2}} \sup \left[\sum_{j=1}^{n} |x_{p_j} - x_{p_{j+1}}|^2 \right]^{1/2},$$

and

$$\nu_5(x) = \left[\frac{\nu_1(x)^2 + \nu_3(x)^2}{3} \right]^{1/2}.$$

James [**192**] showed that the space (J, ν_1) is isometric with its second conjugate, but is not reflexive. Indeed, the canonical image of J in its second dual is of codimension 1. In an earlier paper [**191**], James introduced ν_2.

In 1986, Semenev and Skorik [**340**] showed that the only surjective isometries on (J, ν_3) were trivial ones, except that one could permute the first two coordinates. They also showed that for the "less natural" norm ν_5, the only surjective isometries were trivial. Their results were good for both real and complex scalars. Around the same time, Bellenot [**35**] proved that the space real J with the norm ν_4 has only trivial surjective isometries, and by means of a more general proposition, he showed the existence of separable space X with nonseparable X^{**} having only trivial isometries. He first presented a version of these results at a conference in Kent, Ohio, in 1985. In a long paper concerned with quasi-reflexive spaces which are isometric to their biduals, Sersouri [**342**] showed that the original quasi-reflexive space of James with the norm ν_1 has only trivial isometries.

There is another space on which the isometries have been shown to be essentially trivial. The space, called Tsirelson's space (see [**85**, p. 28]), has trivial isometries, except that the first two components may be permuted. We discussed this space in the notes section of Chapter 9.

Minimal Norms. The construction in this section is due to Jarosz [**198**], and it holds for either real or complex spaces. In the last section of the cited paper, Jarosz discusses the interesting question of when a given group is isomorphic to the group of isometries on a Banach space.

In 1986, Bellenot [**34**] proved that each separable real Banach space has an equivalent minimal norm. He used the notion of *local uniform convexity* [**104**], and the construction was based on what Bellenot called "pimples," a way of decreasing the norm so that the unit ball has two "cones" added. He also notes that the equivalent norms with trivial isometries are dense in the collection of equivalent norms, since equivalent local uniformly convex norms are dense.

Theorem 12.3.5 is due to Wood [**386**, pp. 185-186]. This result does not hold in the complex case, since there exists a compact space E with no non-trivial homeomorphisms for which $C(E, \mathbb{C})$ is maximal [**386**].

Maximal Norms and Forms of Transitivity. The first published results concerning the weaker forms of transitivity were by Rolewicz [**324**], including the proof that a convex transitive norm must be maximal. The proof we have given for Theorem 12.4.1 is essentially that of Cowie, who also introduced the idea of uniquely maximal and proved it was equivalent to convex transitivity [**100**], [**101**]. If X is finite-dimensional, the four types of transitivity are all equivalent and equivalent to X being a Hilbert space. This follows from a result of Auerbach that the group of isometries in this case are unitary operators with respect to some inner product. (See [**324**, Theorem IX 5.1]. The examples of transitive, non-Hilbert space mentioned were also given in [**324**].

Theorem 12.4.3 and its proof are borrowed from both [**324**] and the paper by Greim, Jamison, and Kaminska, who proved a more general version [**158**]. In fact, they also show that an $L^p(\mu)$ space is almost transitive if and only if the measure μ is *homogeneous*. For a finite measure algebra (Ω, Σ, μ), μ is said to be homogeneous if for every $A \in \Sigma$, the measure algebra $(\Sigma|A)/(\mu|A)$ is Boolean isomorphic to Σ/μ (where the quotients refer to the algebras modulo the null sets). The notion is extended to arbitrary positive measure which is then said to be homogeneous if its restriction to every set of finite measure has a homogeneous measure algebra. A number of results concerning almost transitivity in various settings, including Orlicz spaces and $C(K, X)$ spaces, are given in [**158**]. In the compact case, it is easy to see that the uniform norm on $C(K)$ is not AT unless K is a singleton, and Wood [**386**] conjectured that the same thing is true for $C_0(K)$, where K is locally compact. He proved the conjecture to be true when K contains a compact connected set with nonempty interior, or contains a compact set that is both open and closed. Greim and Rajagopolan [**160**] have shown the conjecture to be true for $C_0(K, \mathbb{R})$, but it remained open in the complex case for many years. In 2002, a counterexample was found by Rambla [**307**] (and in 2003, independently by Kawamura [**216**]).

An interesting general result attributed to Lusky [**265**] is this.

12.5.1. THEOREM. *(Lusky) Every Banach space X can be isometrically regarded as a 1-complemented subspace of an almost transitive Banach space having the same density character as X.*

A proof of this version of the theorem is given in the very nice survey paper of Becerra-Guerrero and Rodiguez-Palacios [**24**, Theorem 2.14]. The previously referenced paper [**158**] contains a variety of results on almost transitivity, and a further study of Wood's conjecture involved with such topics as B^*-algebras and superreflexivity is found in [**59**]. B. Randrianantoanina has proved the following theorem [**314**].

12.5.2. THEOREM. *(Randrianantoanina) If X is an almost transitive Banach space, which contains a 1-codimensional, 1-complemented subspace $Z \subset X$, then X is isometric to a Hilbert space.*

Recently, J. Talponen has generalized this theorem [**364**] and has also treated the question with some affirmative results concerning convex transitivity [**363**].

The argument showing that the natural norm on $C([0,1],\mathbb{C})$ is not convex transitive made use of Theorem 12.4.2. It is also easy to show that the definition of CT is not satisfied. For if f is continuous on $[0,1]$ with $f(0) = f(1) = 0$, and T is an isometry on $C[0,1]$, then $Tf(0) = Tf(1) = 0$, so that the convex hull of the orbit $\mathcal{G}(f)$ cannot be dense in the unit ball of $C[0,1]$. The examples of spaces with convex transitive norms were first given by Rolewicz [**324**]. It can be shown that $C_0(0,1)$ is convex transitive. Wood [**386**] has given necessary and sufficient conditions for $C_0(K)$ to be convex transitive.

12.5.3. THEOREM. *(Wood)*

(i) *If K is a locally compact Hausdorff space, then $C_0(K,\mathbb{C})$ is convex transitive if and only if, for every positive measure μ on K with $\|\mu\| = 1$ and $t \in K$, there is a net $\{\varphi_\alpha\}$ of homeomorphisms of K such that $\mu \circ \varphi_\alpha \to \psi_t$ in the weak*-topology.*

(ii) *If K is a locally compact Hausdorff space, then $C_0(K,\mathbb{R})$ is convex transitive if and only if K is totally disconnected and, for every positive measure μ on K with $\|\mu\| = 1$ and $t \in K$, there is a net $\{\varphi_\alpha\}$ of homeomorphisms of K such that $\mu \circ \varphi_\alpha \to \psi_t$ in the weak*-topology.*

The notion of *strongly maximal* was introduced by Becerra-Guerrero and Rodriguez-Palacios [**24**]. A norm on X is strongly maximal if there is no continuous norm on X whose group of surjective isometries strictly enlarges $\mathcal{G}$. To see that the condition is weaker than convex transitivity, suppose the given norm $\|\cdot\|$ is uniquely maximal, and there is a continuous norm ν whose isometry group contains $\mathcal{G}$. Then the norm ρ defined by $\rho(x) = \|x\| + \nu(x)$ is equivalent to $\|\cdot\|$ and its isometry group contains $\mathcal{G}$. By the unique maximality, ρ (and therefore, also ν) is a multiple of $\|\cdot\|$. In [**24**] it is shown that c_0 is strongly maximal, while ℓ^p is not for $1 \le p < 2$. The ℓ^p spaces are maximal, however, as is any symmetric space [**324**, Section IX.8]. Since c_0 is not convex transitive, these examples show that the notion of strongly maximal lies strictly between convex transitivity and maximality.

Kalton and Wood [**212**, Theorem 6.4] proved that a convex transitive Banach space that has a nonzero Hermitian element must be a Hilbert space. A theorem in [**23**] characterizes convex transitive spaces X in terms of the existence of a non-nowhere dense subset of $S(X)$ consisting of *big* points. (An element of the unit sphere is a big point if the convex hull of its orbit by $\mathcal{G}$ is dense in the unit ball.)

The question of maximality for the $C(K)$ spaces was not settled by Rolewicz, and he asked whether the space $C[0,1]$ was maximal. The question was addressed by Kalton and Wood [**212**] and resulted in Lemma 12.4.4

and Theorem 12.4.5. As Wood has said [**387**], it is natural to guess that the larger the group of homeomorphisms of K, the more likely it is that $C(K)$ is maximal. However, the example of a compact space E with no nontrivial homeomorphisms (and, hence, a trivial isometry group) for which $C(E, \mathbb{C})$ is maximal [**386**] runs contrary to that. (We mentioned this example at the end of the previous subsection.) In fact, Wood made the following conjecture.

12.5.4. CONJECTURE. *If K is a locally compact connected Hausdorff space, then $C_0(K, \mathbb{C})$ is maximal if and only if the set of all homeomorphisms of K are either trivial or infinite.*

The statement above is as it was worded by Lin [**258**], who showed that the condition stated above is necessary for maximality, but not sufficient. Lin [**256**] has also shown the maximality of the norm in certain rearrangement-invariant spaces.

Partington's proof [**297**] that the norm on the real space $C[0, 1]$ is not maximal answered a question left by Wood [**386**].

A question which is still open and asked first by Wood in 1982 [**386**] (see also [**24**]) is this: Does every Banach space have an equivalent maximal norm? A variation on this would be to ask if there is an equivalent maximal norm for which the original isometries are still isometries [**389**].

The property of maximality for a Banach space X is equivalent to $\mathcal{G}(X)$ being a maximal bounded group in $\mathcal{L}(X)$ and so is a condition on the algebra $\mathcal{L}(X)$. An element u of a unital Banach algebra $\mathcal{A}$ is said to be *unitary* if u is invertible and $\|u\| = \|u^{-1}\| = 1$. The algebra A is then said to be *(algebra) maximal* if the set of unitary elements is a maximal bounded group in A. This notion was introduced by Cowie in 1981 [**100**], who also studied analogues of convex transitive and uniquely maximal for algebras. These ideas were also discussed independently later by Hansen and Kadison [**174**]. Good references for the results for algebras can be found in [**25**], [**388**], and [**390**].

As we mentioned at the beginning of these notes, the subject we have been discussing essentially began with the results announced by Pelczynski and Rolewicz at the International Mathematical Congress in 1962, and our references to them have been to the book of Rolewicz [**324**]. Kalton and Wood then gave impetus to the study in [**212**]. Wood kept the ball rolling with a series of papers through the years, including [**386**], [**387**], [**388**], and important contributions from his student Cowie [**100**], [**101**]. We should mention the following papers, which are especially pertinent to the study of almost transitivity, convex transitivity, maximality, and connections with Banach algebras and Jordan structure: [**22**], [**23**], [**59**], [**58**], and [**56**]. We have left many important papers and ideas unmentioned here, and the reader should consult the survey papers [**24**] and [**57**] for a guide to further study.

We close by listing two older papers that are appropriate to this chapter: [**165**] and [**385**].

CHAPTER 13

Epilogue

In this closing chapter we intend to touch on a number of topics that are related to the study of isometries in some way, or isometry results that we have not taken time to consider in the previous chapters. There will be few proofs here, but rather, a brief explanation of the subject, a few important results, and a guide for further investigation. The topics are covered in no particular order, and we may rely on only two or three principal sources, leaving it to the reader to find other contributors from the reference lists.

13.1. Reflexivity of the Isometry Group

For a Banach space X and a subset $\mathcal{S}$ of $\mathcal{L}(X)$ let

$$ref_{al}(\mathcal{S}) = \{T \in \mathcal{L}(X) : Tx \in \mathcal{S}x \text{ for all } x \in X\},$$

$$ref_{to}(\mathcal{S}) = \{T \in \mathcal{L}(X) : Tx \in \overline{\mathcal{S}x} \text{ for all } x \in X\},$$

where $\mathcal{S}x = \{Rx : R \in \mathcal{S}\}$. The set $\mathcal{S}$ is said to be *algebraically reflexive* if $ref_{al}(\mathcal{S}) = \mathcal{S}$ and *topologically reflexive* if $ref_{to}(\mathcal{S}) = \mathcal{S}$. We are interested in the case where $\mathcal{S}$ is the isometry group $\mathcal{G}(X)$, and we will say that X is *iso-reflexive* if $\mathcal{G}(X)$ is algebraically reflexive. Similarly, we will call X *topogically iso-reflexive* if $\mathcal{G}(X)$ is topologically reflexive. We have taken these definitions from the papers of Cabello-Sanchez and Molnar [**60**] and Molnar and Zalar [**284**] and recommend these papers and their bibliographies for those who want to pursue this subject. The earliest results of this type, which involved derivation algebras, should probably be attributed to Kadison [**210**], Larson [**230**], and Larson and Sourour [**231**]. An early consideration of the isometry group was given by Molnar [**282**].

We will refer to an element of $ref_{al}(\mathcal{G}(X))$ as a *local isometry* (or more precisely, a local surjective isometry), and an element of $ref_{to}(\mathcal{G}(X))$ as an *approximate local isometry.* Hence X is iso-reflexive if and only if every local surjective isometry is a surjective isometry. Since a local isometry is always an isometry, it is clear that any finite-dimensional Banach space is iso-reflexive. Furthermore, any space which has nonsurjective isometries and satisfies the transitive property (discussed in Chapter 12) will fail to be iso-reflexive, since in this case, any isometry is a local isometry. Hence, we have the following easy result.

13.1.1. Proposition. *A Hilbert space H is iso-reflexive if and only if it is finite-dimensional.*

Similarly, it is straightforward to show that if X is a space with an almost transitive norm, then any isometry is an approximate local isometry so that if X has nonsurjective isometries, X cannot be topologically iso-reflexive. It is a consequence of this remark, for example, that $L^p[0,1]$ is not topologically iso-reflexive since the norm is almost transitive by Theorem 12.4.3 from Chapter 12. In fact, this space is not iso-reflexive either.

Topological iso-reflexivity is a stronger property than iso-reflexivity, and it might seem that the topological condition is rare. However, we have the following interesting fact.

13.1.2. THEOREM. *If X is any Banach space, then X admits a renorming whose isometry group is topologically reflexive.*

PROOF. First we observe that if X is a space with a minimal norm, then X is topologically iso-reflexive. (Recall from Chapter 12, that a space with a minimal norm has an isometry group consisting of modulus 1 multiples of the identity.) Hence, if T is an approximate local isometry and x has norm 1, then for every positive integer n there is some modulus 1 scalar λ_n such that $\|Tx - \lambda_n x\| < 1/n$. It follows that there is some λ with $|\lambda| = 1$ such that $Tx = \lambda x$ so that an approximate local isometry must be a local isometry. But an isometry T that is not surjective cannot be a local isometry, since if y is not in the range of T, we cannot have $Ty = Uy$ for any trivial isometry U. Hence any approximate local isometry must be surjective, and X is topologically iso-reflexive. The conclusion of the theorem now follows from Theorem 12.3.4 from Chapter 12. □

This theorem was given in [**60**] as a negative answer to a conjecture stated in [**284**]. We also see from the proof that any space with minimal norm is topologically iso-reflexive, so that the spaces with minimal norm discussed in Chapter 12 satisfy this property.

Suppose T is a local isometry on the space $X = \ell^p$, where $p \neq 2, 1 \leq p < \infty$. Recall that every surjective isometry is of the form

$$\sum x_n e_n \to \sum \alpha_n x_n e_{\varphi(n)},$$

where $|\alpha_n| = 1$ and φ is a permutation of $\mathbb{N}$. Hence there is an injective mapping ψ on $\mathbb{N}$ so that

$$Te_n = \beta_n e_{\psi(n)},$$

where $|\beta_n| = 1$. If we choose $x = \sum x_n e_n$ such that the x_n's are positive and distinct from each other, there is a surjective isometry U such that $Tx = Ux$, and we have

$$\sum \beta_n x_n e_{\psi(n)} = \sum \alpha_n x_n e_{\varphi(n)},$$

for a permutation φ. It follows that $\psi = \varphi$, and so T is actually surjective. We conclude that X is iso-reflexive. Basically the same argument shows that any Banach space (not ℓ^2) which has a symmetric basis is iso-reflexive.

In the next theorem, we collect some of the results that are known about specific spaces. The list is not exhaustive, nor are the results necessarily given in the most generality possible. The proofs can be found in [**60**] and [**284**].

13.1.3. Theorem.

(i) *The following spaces are topologically iso-reflexive:*
  (a) *The Hardy spaces H^p on the disk, for $p \neq 2$;*
  (b) *The disk algebra $A(D)$;*
  (c) *The bounded operators $\mathcal{L}(H)$ on a Hilbert space. H* [**282**]

(ii) *The spaces below are iso-reflexive but not topologically iso-reflexive:*
  (a) *The Schatten p-classes, $\mathcal{C}_p(H)$ for $p \neq 2$. More generally, this is true for symmetric norm ideals not isomorphic to the Hilbert-Schmidt class;*
  (b) *The ℓ^p spaces for $1 \leq p \leq \infty$ and c_0;*
  (c) *$C(K)$, where K is a first countable compact Hausdorff space.*

(iii) *The following spaces are not iso-reflexive:*
  (a) *$L^p(\mu)$ spaces, where μ is purely nonatomic and homogeneous;*
  (b) *$c(L)$, where L is an uncountable index set.*

13.1.4. Corollary. *The spaces H^p $(p \neq 2)$, $A(D)$, and $\mathcal{L}(H)$ are not almost transitive.*

This corollary holds, of course, because, as we observed earlier, if the spaces were almost transitive, then every isometry would be an approximate local isometry and hence surjective by the topological iso-reflexivity. Perhaps the facts given in this corollary are known, but we have not seen them mentioned in print.

Iso-reflexivity is clearly preserved by isometric isomorphisms, but is not inherited by closed subspaces, since every separable Banach space can be embedded inside of $C[0,1]$, which is iso-reflexive. It is shown in [**60**], in connection with a question raised in [**284**], that there is no relationship between iso-reflexivity of a space and its dual. Indeed, there exist spaces for which the space is iso-reflexive and its dual is not, and viceversa.

A related concept of *2-locality* was introduced by Semrl [**375**] and applied to isometries by Molnar [**283**]. A mapping $\phi : X \to X$ (no linearity is assumed) is called a 2-local isometry of X if for every $x, y \in X$ there is a surjective isometry $T_{x,y}$ of X, depending on x and y, such that $T_{x,y}(x) = \phi(x)$ and $T_{x,y}(y) = \phi(y)$. Molnar [**283**] showed that every 2-local isometry on $\mathcal{L}(H)$ is linear and so a surjective isometry. Gyory [**172**] has obtained results on 2-local isometries for $C_0(K)$.

We close this section with a problem posed in [**60**] and still open so far as we know. If K is a locally compact Hausdorff space and $C_0(K)$ is separable, must it be iso-reflexive?

13.2. Adjoint Abelian Operators

The notion of Hermitian operator on a Banach space has been of importance in certain contexts in finding the description of isometries. The definition in terms of a semi-inner product $[\cdot,\cdot]$ on a Banach space X as an operator T for which $[Tx,x]$ is real for each $x \in X$ is obviously patterned after one of the properties of a self-adjoint or Hermitian operator on a Hilbert space. The "self-adjoint" property leads us to ask about operators for which $[Tx,y] = [x,Ty]$ for all x, y in the Banach space. The study of such operators was introduced by Stampfli in 1969 [**357**] who called them *adjoint abelian*.

13.2.1. DEFINITION. *An operator T on a Banach space X is said to be* adjoint abelian *if there exists a semi-inner product $[\cdot,\cdot]$ compatible with the norm of X so that*

$$[Tx,y] = [x,Ty] \ \textit{for all}\ x,y \in X.$$

Since a semi-inner product is given by a duality map, an operator T is adjoint abelian if and only if there is a duality map ϕ from X to X^* so that

$$\phi T(y) = T^*\phi(y)$$

for every y, which is the form in which Stampfli originally gave the definition, and the clear reason for the name he attached.

If we consider the s.i.p. on $\ell^p(2)$, for $1 < p < \infty, p \neq 2$, given by

$$[x,y] = x(1)\overline{y(1)}\left(\frac{|y(1)|}{\|y\|}\right)^{p-2} + x(2)\overline{y(2)}\left(\frac{|y(2)|}{\|y\|}\right)^{p-2}$$

then the operators given by the matrices (with respect to standard bases)

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

are, respectively, adjoint abelian but not Hermitian, and Hermitian but not adjoint abelian. Hence the notions separate on non-Hilbert spaces and one might ask if this somehow characterizes Hilbert spaces. We will see below that this is not the case.

Stampfli [**357**] established several general facts about adjoint abelian operators, including the one that an operator on a weakly complete Banach space that commutes with a nontrivial adjoint abelian operator must have a proper invariant subspace. In this same paper Stampfli asked whether every adjoint abelian operator on a weakly complete space is necessarily a scalar operator, and this question remains open. (See also Dunford and Schwartz [**116**, pp. 2105-2106].)

Let us consider an ℓ^p sum of Banach spaces X_n,

$$X = \left(\sum_n \oplus X_n\right)_p,$$

where $1 < p < \infty, p \neq 2$. A semi-inner product compatible with the norm ν on X is given by

$$[x,y] = \sum_{n=1}^{\infty} [x(k), y(k)]_k \left(\frac{\nu_k(y(k))}{\nu(y)} \right)^{p-2}, \tag{121}$$

where ν_k denotes the norm on X_k and $[\cdot,\cdot]_k$ is a s.i.p. on X_k compatible with ν_k.

13.2.2. THEOREM. *An operator A is an adjoint abelian operator on the space $X = \left(\sum_n \oplus X_n\right)_p$ if and only if $A = \lambda U$, where U is an isometry on X with $U^2 = I$ and λ is real.*

PROOF. Let us indicate a sketch of how this argument goes. Details may be found in [**127**] and [**134**].

If A is a.a., then $A^2 = T$ is both Hermitian and adjoint abelian, and so by [**134**, Theorem 3.3], T may be represented as a diagonal matrix operator $T = (T_{nn})$, where T_{nn} is both Hermitian and adjoint abelian on X_n. Using the form given by (121), and the fact that $[Tx, y] = [x, Ty]$, we obtain

$$0 = \sum_{n=1}^{\infty} [T_{nn}x(n), y(n)] \left[\left(\frac{\nu_n(y(n)}{\nu(y)} \right)^{p-2} - \left(\frac{\nu_n(T_{nn}y(n))}{\nu(Ty)} \right)^{p-2} \right]. \tag{122}$$

It is not difficult to show that $T_{nn} \neq 0$ for each n. Now suppose $T_{kk}y(k) = 0$ and choose $x(1), y(1) \in X_1$ such that $[T_{11}x(1), y(1)] \neq 0$ and let $x = x(1), y = y(1) + y(k)$. Upon substitution into (122), we have

$$0 = [T_{11}x(1), y(1)] \left[\left(\frac{\nu_1(y(1))}{\nu(y)} \right)^{p-2} - 1 \right].$$

It follows that $y(k) = 0$. Choosing an appropriate x in (122) we can show that

$$\frac{\nu_k(T_{kk}y(k))}{\nu_k(y(k))} = \frac{\nu(Ty)}{\nu(y)}$$

for any y with $y_k \neq 0$. From this it follows that $\frac{\nu(Ty}{\nu(y)}$ is constant for $y \neq 0$, so that there exists $\lambda > 0$ such that

$$\nu(Ty) = \lambda \nu(y) \text{ for all } y.$$

Hence, $\lambda^{-1}T$ is an isometry. The desired conclusion follows from the theorems stated below. □

13.2.3. THEOREM. *If T is adjoint abelian on X and $T^2 = \lambda I$ for some $\lambda > 0$, then T is a positive multiple of an isometry.*

PROOF. If $T^2 = \lambda I$, let $r = \sqrt{\lambda}$. Then

$$\begin{aligned} \|(r^{-1}T)(x)\|^2 &= [r^{-1}Tx, r^{-1}Tx] = (r^{-1})^2[Tx, Tx] \\ &= \lambda^{-1}[T^2x, x] = [x, x] = \|x\|^2. \end{aligned}$$

□

13.2.4. Theorem. *Let U be an isometry on a Banach space X and λ a real number. Then $T = \lambda U$ is adjoint abelian if and only if $U^2 = I$.*

Proof. If $T = \lambda U$ is a.a., we have

$$[(\lambda U)^2 x, x] = [\lambda U x, \lambda U x] = \lambda^2 [x, x],$$

so that $[(U^2 - I)x, x] = 0$ for every x. It follows from a theorem of Lumer [**264**, Theorem 5] that $U^2 = I$.

On the other hand, if U is an isometry with $U^2 = I$, by a theorem of Koehler and Rosenthal [**222**] there exists a compatible s.i.p. $[\cdot, \cdot]$ such that

$$[Ux, Uy] = [x, y]$$

for all x, y. Hence,

$$[\lambda U x, y] = [\lambda U x, U^2 y] = [\lambda x, U y] = [x, \lambda U y].$$

□

Stampfli [**356**] has shown that an operator is scalar if its square is an invertible scalar operator. This leads to the following corollary.

13.2.5. Corollary. *Every adjoint abelian operator which is a multiple of an isometry is necessarily a scalar operator.*

Coupling the above results with known characterizations of isometries, it is possible to give complete descriptions of adjoint abelian operators on some specific Banach spaces.

We have shown [**129**] that $C(K)$, for K compact and metric, and also $L^p(\Omega, \Sigma, \mu)$, for σ-finite measures and $1 < p < \infty, p \neq 2$, satisfy the hypotheses of Theorem 13.2.3. The corresponding descriptions of the operators use the fact that the associated isometries are reflections, so that the underlying homeomorphisms and set isomorphisms must have the "self-inverse" property. In fact, we can get the same result for a Bochner space $L^p(\mu, X)$ under certain conditions when X is a Banach space with trivial Hermitians [**134**], and also for $L^p(\mu, H)$, where H is a separable Hilbert space [**130**]. Grzesiak [**169**] has shown that all adjoint abelian operators on smooth Orlicz spaces are positive multiples of an isometry. We note that in all of these cases, the adjoint abelian operators are scalar operators.

Not every adjoint abelian operator is a multiple of an isometry. For example, the operator A given by the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

on the two-dimensional modulared space X with s.i.p. given by

$$[x, y] = \|y\| \left(\frac{x(1)\overline{y(1)}}{2\|y\|} + x(2) sgn y(2) \right)$$

is adjoint abelian, but not a multiple of an isometry. Note also, that

$$A^2 + \lambda I = \begin{bmatrix} 1+\lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

is not adjoint abelian for any $\lambda \neq 0$, which gives a negative answer to a conjecture of Istratescu [**189**] (see also [**295**]). The above example along with some other results can be found in [**131**].

We wish to look at one last example. For $1 < p < \infty, p \neq 2$, and $\alpha > 0$, let $s_p(\alpha)$ denote the space of all sequences $x = (x(j))$ for which

$$\|x\| = \left(\sum_{j=1}^{\infty} |x_j|^p + \alpha |x_{j+1} - x_j|^p \right)^{1/p}$$

is finite. Then $s_p(\alpha)$ is a Banach space which is isomorphic to ℓ^p. For $\alpha \neq 1$, the isometries on $s_p(\alpha)$ are trivial and the Hermitian operators are real multiples of the identity [**133**]. Since the adjoint abelian operators are also real multiples of the identity, we see that there do exist non-Hilbert spaces on which the Hermitian and adjoint abelian operators are the same. (See also [**83**].)

Some more recent papers involving adjoint abelian operators include [**151**], [**260**], and [**341**].

13.3. Almost Isometries

Banach [**18**, p. 242] defined two isomorphic Banach spaces X, Y to be *almost isometric* if the number

$$(X, Y) = \inf\{\log \|T\|\|T^{-1}\| : T \text{ is an isomorphism from } X \text{ onto } Y\}$$

is equal to zero. He noted that isometric spaces are almost isometric and that the converse holds for finite-dimensional spaces. In particular, he asked whether the sequence spaces c and c_0 are almost isometric. It has been common to drop the logarithm in the above definition and consider

$$d(X, Y) = \inf\{\|T\|\|T^{-1}\| : T \text{ is an isomorphism from } X \text{ onto } Y\},$$

which is known as the Banach-Mazur distance from X to Y. Thus spaces are almost isometric if their Banach-Mazur distance is 1.

In 1966, Cambern [**61**] answered the question of Banach by showing that c and c_0 are not almost isometric. In fact, we have the following theorem.

13.3.1. THEOREM. *(Cambern and Amir) If Q and K are locally compact spaces and the Banach-Mazur distance between $C_0(Q)$ and $C_0(K)$ is less than 2, then Q and K are homeomorphic.*

The advertised homeomorphism between K and Q arises from the fact for any pair, $x \in Q$, $y \in K$, and isomorphism T from $C_0(Q)$ onto $C_0(K)$ with

$1 \leq \|T\| < 2$, there are a unique complex number α and $\mu \in C_0(Q)^*$ such that

$$T^*\psi_y = \alpha\psi_x + \mu.$$

Here, as usual, ψ_y refers to the point-mass measure at y, or equivalently, the evaluation functional at y. The fact that this pairing between y and x is actually a homeomorphism requires considerable argument, which we are omitting.

Since c and c_0 are not isometric, we see that their Banach-Mazur distance is at least 2. Cambern remarks in [**61**] that this fact had been proved also by Pelczynski, but not published.

Theorem 13.3.1 is a generalization of the weak form of the Banach-Stone theorem, and it implies that $C_{0)}(Q)$ and $C_0(K)$ are actually isometric. Therefore, in this setting, the notions of isometric and almost isometric are the same. In [**61**], Cambern assumed that Q, K were first countable, but removed that condition in [**62**]. Amir [**8**] established the result independently for Q, K compact and for real-valued functions. He gave several nice examples, and asked whether there exist nonhomeomorphic compact Hausdorff spaces Q, K such that the Banach-Mazur distance from $C(Q)$ to $C(K)$ is less than 3. Gordon [**148**] showed the answer was no if Q and K are countably compact. In 1970, Cambern [**63**] gave an example of a compact Q and noncompact K for which the Banach-Mazur distance from $C(Q)$ to $C_0(K)$ is 2. Cohen [**98**] showed there was such an example where both Q and K are compact.

In 1948, Myers [**287**] had proved that if a completely regular subspace M of $C(Q)$ is isometrically isomorphic to a completely regular subspace N of $C(K)$, then Q and K must be homeomorphic. Cengiz [**86**] has essentially extended both Myers' and Cambern's results with the following theorem.

13.3.2. THEOREM. *(Cengiz) Let Q and K be locally compact spaces and let M and N be extremely regular linear subspaces of $C_0(Q)$ and $C_0(K)$, respectively. If there is a linear isomorphism T of M onto N with $\|T\|\|T^{-1}\| < 2$, then Q and K are homeomorphic.*

There is a brief discussion of this result (along with definitions) also in the notes section of Chapter 2. Contributions along these lines were also made by Rochberg [**321, 322, 323**].

Turning to the vector-valued case, Cambern in 1976 [**67**] established Theorem 13.3.1 for the spaces $C(Q, E)$ and $C(K, E)$, where E is a finite-dimensional Hilbert space, assuming the Banach-Mazur distance is less than $\sqrt{2}$. Later he was able to do this when E is uniformly convex, but the number $\sqrt{2}$ is replaced by a number that is less than 2, but involves the modulus of convexity associated with E [**76**]. Jarosz [**195**] has also contributed a result for a Banach space whose dual space satisfies certain conditions.

In analogy with the isometric Banach-Stone property (see Chapter 7), Behrends [**30**] has introduced the notion of the *isomorphic Banach-Stone property*.

13.3.3. Definition. *A Banach space X is said to have the* isomorphic Banach-Stone property *if there is a $\delta > 0$ such that the following holds: whenever Q and K are locally compact Hausdorff spaces such that there exists an isomorphism $T : C_0(Q, X) \to C_0(K, X)$ with $\|T\|\|T^{-1}\| \leq 1 + \delta$, then Q and K are homeomorphic.*

Hence, by the result of Cambern mentioned above, a uniformly convex space has the isomorphic Banach-Stone property. As pointed out in [**30**], little is known about the range of the possible constants δ for a space X with the isomorphic Banach-Stone property, except for the one-dimensional case for X, where the $\delta < 1$ are admissible.

13.3.4. Definition. [**30**] *A Banach space X is said to have the* strong isomorphic Banach-Stone property *if for every $\epsilon > 0$, there is a $\delta > 0$ such that the following holds: whenever Q, K, and T are given as in Definition 13.3.3, then there exist a homeomorphism $\varphi : Q \to K$ and a familty $(T_s)_{s \in Q}$ of isomorphisms on X such that*

$$\|T_s\|\|T_s^{-1}\| \leq 1 + \epsilon \quad \textit{and}$$

$$\|(TF)(\varphi(s)) - T_s[F(s)]\| \leq \epsilon$$

for all $s \in Q$ and all $F \in C_0(Q, X)$ with $\|F\| = 1$.

Behrends and Cambern [**31**] have established a theorem concerning the isomorphic Banach-Stone property which they indicate includes "all isomorphic Banach-Stone theorems existing in the literature." A different approach is followed in [**30**], and while the δ's obtained are usually smaller, spaces with the strong property of 13.3.4 are considered.

It seems that Benyamini, in part of his 1975 Ph.D. thesis [**36**], was the first to talk about almost isometries on the classical L^p spaces. He showed there that (in the separable case) for each p, $1 \leq p \leq \infty, p \neq 2$, there is a constant k_p so that if the Banach-Mazur distance between $L^p(\mu)$ and $L^p(\nu)$ is less than k_p, then the spaces are isometric. Cambern [**73**] established this result for σ-finite measure spaces and $p = 1$ or $p = \infty$, and where $k_1 = 2$. The same result was extended to Bochner spaces $L^p(\mu, X)$, where X is a Hilbert space and the Banach-Mazur distance is less than $3/(2\sqrt{2})$ [**77**]. Cambern, Jarosz, and Wodinski [**82**] extended the Benyamini result to spaces $L^p(\mu, X), L^p(\nu, Y)$, where X, Y are Banach spaces satisfying certain properties, and where the needed Banach-Mazur distance depends on p and the spaces X, Y.

Benyamini [**37**] actually showed that for Q compact metric, if T is an isomorphism from $C(Q)$ into $C(K)$ with $\|T\|\|T^{-1}\| < 1 + \epsilon$, then there is an isometry W with $\|T - W\| < 3\epsilon$. However, he gave an example showing that for any $\epsilon > 0$, there is an isomorphism satisfying the above, but that $\|T - W\| > 2$ for all isometries. (See also [**38**].) Alspach [**7**] showed that given an isomorphism T between $L^p(\mu)$ and $L^p(\nu)$, then if $\|T\|\|T^{-1}\|$ is sufficiently small, T is close to an isometry. As observed in [**82**], Alspach's result cannot be extended to the vector case in general, since Jarosz [**196**, p. 93] showed there are Banach spaces X which for any $\epsilon > 0$ admit isomorphisms T with $\|T\|\|T^{-1}\| < 1+\epsilon$, but $\|T - W\| \geq 2 - \epsilon$ for any isometry W on X.

The notion of finding an isometry that is close to an isomorphism suggests a slightly different approach to the idea of almost isometry. Hyers and Ulam [**186**] called a transformation T from one Banach space X to another an *ϵ-isometry* if

$$|\|x-y\| - \|Tx - Ty\|| \leq \epsilon \text{ for all } x, y \in X.$$

The question then raised was whether, given two Banach spaces X, Y, there exists some constant k such that for $\epsilon > 0$ and a surjective ϵ-isometry $T : X \to Y$,that there is an isometry U from X to Y such that $\|Tx - Ux\| \leq k\epsilon$ for all $x \in X$. A considerable amount of work on this and similar questions was carried out by Hyers and Ulam [**186, 187**], D. Bourgin [**50, 51, 52**], R. Bourgin [**53**], Gruber [**166**], Gevirtz [**143**], and Lovblom [**262, 263**]. A brief but more enlightening discussion of these matters may be found in [**136**]. A couple of more recent papers which treat almost isometries include [**294**] (related to Lovblom's work) and [**111**] (the most recent in a string of papers).

13.4. Distance One Preserving Maps

An old question of Aleksandrov asks under what conditions a mapping on a metric space which preserves unit distance must necessarily be an isometry [**318**]. We have a short discussion of this question in [**136**], but we intend to say no more here than that the reader should consult the writings of T. Rassias, and in particular the book of Hyers, Isac, and Rassias [**185**].

A similar problem, sometimes referred to as *Tingley's problem* [**120**], is this: If T maps the unit sphere of a real Banach space X onto the unit sphere of a real Banach space Y and preserves distance, can T be extended to a linear isometry of X onto Y [**367**]? This is related to the early paper of Mazur and Ulam [**279**] and also work of Mankiewicz [**266**]. The problem has received attention of late, of which the first citation above as well as [**391**] and [**121**] are good examples. Another interesting paper from a bit earlier is [**379**].

Still another variation on this theme is the study of *diameter-preserving maps*. We mention three papers that will introduce the reader to this topic, [**19**], [**173**], and [**317**].

13.5. Spectral Isometries

A linear mapping T between two unital C^*-algebras A, B is called a *spectral isometry* if the spectral radius of Ta equals the spectral radius of a for each $a \in A$. Recall that by Kadison's theorem (Theorem 6.1.1 of Chapter 6) an isometry between A and B is the product of a unitary operator and a Jordan*-isomorphism. A Jordan*-isomorphism (and hence an isometry) necessarily preserves the spectral radius and is a spectral isometry. Mathieu [**275**] poses the following interesting conjecture.

13.5.1. CONJECTURE. *Every unital surjective spectral isometry between unital C^*-algebras is a Jordan-isomorphism.*

Everything discussed in this section comes from one source, the paper of Mathieu [**275**] cited above.

In what follows we will assume that T is a unital surjective spectral isometry between unital C^*-algebras A and B. If either A or B is commutative, then it is known that T is a multiplicative isomorphism, and we have an isometry. In fact, every surjective spectral isometry restricts to an isomorphism of the centers of general C^*-algebras. Unital isometries are self-adjoint, and self-adjoint unital surjective spectral isometries are isometries. Thus a goal in the study of spectral isometries as in the conjecture, as Mathieu puts it, is to find a non-self-adjoint version of Kadison's theorem. We conclude our discussion of this topic by stating the following from Mathieu's paper.

13.5.2. THEOREM. *Let $T : A \to B$ be a unital surjective spectral isometry between the unital C^*-algebras A and B. If either*

(i) *A is a von Neumann algebra without direct summand of type II_1*

or

(ii) *A is a simple C^*-algebra with real rank zero and without tracial states,*

then T is a Jordan-isomorphism.

We probably should point out that there have also been studies of transformations that preserve the numerical radius, and [**247**] will provide a window into that topic.

13.6. Isometric Equivalence

The notion of unitary equivalence for operators on a Hilbert space is a common topic of interest, but the analagous idea for operators on Banach spaces is less known and studied. If X is a Banach space, we say that two bounded linear operators T, S are *isometrically equivalent* if there exists a surjective isometry U on X such that $UT = SU$. The problem is to identify properties the operators must satisfy in order to be isometrically equivalent. The earliest work along these lines goes back to 1961 in a paper of Kalisch [**211**], in which Kalisch established conditions for the equivalence of certain kernel operators on L^p spaces. He defined and used the term *isometric equivalence* in that paper.

The isometric equivalence of composition operators was addressed many years later by Campbell-Wright [**84**], who proved the following theorem.

13.6.1. THEOREM. *Let ϕ and ψ be analytic maps of the disk. The composition operators C_ϕ and C_ψ are isometrically equivalent, as operators on $H^p(D)$ $(1 \leq p < \infty,\ p \neq 2)$, if and only if there is a real number θ such that*

$$\phi(z) = e^{-i\theta}\psi(e^{i\theta}z)$$

for all z in the unit disk D.

By C_ϕ, of course, is meant the composition operator defined by $C_\phi(f) = f \circ \phi$. Hornor and Jamison [**181**] studied isometric equivalence of composition operators on the Bloch space and $S^p(D)$. (See Chapter 4 for a discussion of these spaces.) Later, they did the same for Hilbert space-valued versions of these same spaces [**182**].

Recently, Jamison [**193**] has considered the isometric equivalence problem on weighted shifts and other types of operators on Banach spaces. This paper is a good one to read for a bit of history of the subject and some good references. We also mention the paper of Gal [**141**], which considers the problem for a variety of matrix operators.

13.7. Potpourri

In this last section it is our intent to list some references that have been left out in the preceding discussions. We are just going to list the numbers (without names), but we will try to group them where possible under some general topic. There will still be many worthy papers omitted, and we urge the reader who does not find what is sought here to check the reference list in [**136**], the reference list in Volume 1, and, of course, MathSciNet under the topic "isometries."

Some interesting papers related to the Banach-Stone theorem are [**4**], [**110**], [**170**], [**199**], [**271**], [**290**], [**343**], [**349**], and [**373**].

Papers concerning isometries of tensor products include [**197**], [**217**], [**220**], and [**333**], while [**3**] and [**117**] involve isometries on lattices.

Next, we list some papers involving groups, [**164**], [**206**], [**214**], and [**237**]; and algebras, [**13**], [**14**], [**96**], [**97**], [**123**], [**140**], [**144**], [**145**], [**190**], [**204**], [**274**], [**283**], [**353**], [**358**], and [**382**].

Following is a number of papers which are concerned with isometries on a variety of special spaces: [**9**], [**107**], [**108**], [**105**], [**112**], [**122**], [**135**], [**167**], [**175**], [**218**], [**261**], [**273**], [**272**], [**276**], [**277**], [**278**], [**304**], [**311**], [**312**], [**316**], [**328**], [**359**], [**368**], [**370**], and [**384**].

Finally, we mention a few other papers pertaining to isometries and general Banach spaces: [**41**], [**124**], [**149**], [**183**], [**184**], [**243**], [**288**], [**352**], and [**376**].

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